Θ dependence of 4D SU(N) gauge theories

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The dependence of QCD, and general 4D SU(N) gauge theories, on the topological θ term is addressed at zero and finite temperature, and in particular in the large-N limit.

General arguments and numerical analyses exploiting the lattice formulation show that it drastically changes across the deconfinement transition. The low-T phase is characterized by a large-N scaling with θ/N as relevant variable, while in the high-T phase the scaling variable is just θ and the free energy is essentially determined by the instanton-gas approximation.

QCD, SU($N = 3$) gauge theory with $N_f = 6$ quarks

$$\mathcal{L}_{\text{Euclidean}} = \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) + \sum_{f=1}^{N_f} \bar{\psi}_f \left(D + m_f\right) \psi_f .$$

Actually, its reduced version $\mathcal{L}_{\text{lite}} = \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) + \sum_{f=u,d} \bar{\psi}_f D \psi_f$ (no free parameters) provides most matter of visible universe

Some flavors are very light, $n_f = 2, 3$, thus chiral ($m_f = 0$) simmetries:

- classically, $\mathrm{U}(n_f)_L \otimes \mathrm{U}(n_f)_R \approx \mathrm{U}(1)_V \otimes \mathrm{U}(1)_A \otimes \mathrm{SU}(n_f)_V \otimes \mathrm{SU}(n_f)_A$
- but the spectrum shows only $\mathrm{U}(1)_V \otimes \mathrm{SU}(n_f)_V$, and quasi-Goldstone bosons associated with SU$(n_f)_A$
- In particular, θ-dependence is connected with the missing U$(1)_A$
- quark masses $m_u \approx m_d/2 \approx 2.5$ Mev $\longrightarrow$ soft breaking

The strong-interaction theory QCD, and general 4D SU(N) gauge theories, have a nontrivial θ dependence

$$\mathcal{L}_{\theta,\mathrm{Euclidean}} = \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) - i\theta \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x),$$

$q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$ is the topological charge density.

$\longrightarrow$ The topological θ term violates parity and time reversal.

$|\theta| \lesssim 10^{-9}$ from experimental bounds on the neutron electric dipole moment: $|d_n| < 3 \times 10^{-26} e\,\mathrm{cm}$, and $d_n \sim \theta e m_\pi^2 / m_n^3 \approx 10^{-16} \theta\, e\,\mathrm{cm}$.

$\longrightarrow$ Nevertheless θ dependence remains an interesting issue,

for example, $U(1)_A$ problem $\rightarrow$ the axial $U(1)_A$ symmetry is not realized in the QCD spectrum, neither explicitly nor as a Goldstone mechanism $(m_{\eta'} > \sqrt{3} m_\pi)$, being violated at quantum level

• θ dependence vanishes in perturbation theory.

• In the semiclassical picture, contributions from classical instanton solutions with $Q = \int d^4x\, q[A_I(x)] \neq 0$ give rise to tunneling between n-vacua, leading to θ vacua: $|\theta\rangle = \sum_n e^{in\theta}|n\rangle$.

• Numerical evidence of a nontrivial θ dependence from the lattice formulation of the theory.

• The $U(1)_A$ charge is not conserved due to the chiral anomaly $\partial_\mu j_5^\mu(x) = i2N_f q(x) \longrightarrow$ explaining the large η' mass.

• At finite T: θ-dep is related to the softening of the $U(1)_A$ breaking,

effective $U(1)_A$ symmetry restoration, T-dep of η' mass, nature of the hadron-to-quarkgluon transition, spectrum of the excitations, etc...

possible evidences from heavy-ion collisions

Plan of the talk

• General scenario for the θ dep from $T = 0$ to high-T, mostly discussed within pure gauge theories

• θ dependence within the large-N framework $\longrightarrow$ large-N scaling expected to hold at $T = 0$

• analytic and periodic θ dependence from instanton-gas approximations $\longrightarrow$ expected to be effective at $T \gg T_c$

• sharp change of θ dependence across the deconfinement transition

• Overview of lattice results for the θ dependence at $T = 0$ and finite T, across the deconfinement transition, supporting the θ dependence

• Monte Carlo simulations in the presence of θ are affected by the sign problem: expansion around $\theta = 0$ and simulations at imaginary θ

• Conclusions and a few more remarks on θ dependence in full QCD

θ dependence of the ground-state energy

$$E(\theta) = -\frac{1}{V_4} \ln \int [dA] \exp \left(- \int d^4x \mathcal{L}_\theta \right)$$

$$\mathcal{L}_\theta = \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) - i\theta q(x), \qquad q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$$

The free energy at finite temperature (Gross, Pisarski, Yaffe, RMP 1981)

$$F(\theta, T) = -\frac{1}{\mathcal{V}_4} \ln \mathrm{Tr} e^{-H/T} = -\frac{1}{\mathcal{V}_4} \ln \int [dA] \exp \left(- \int_0^{1/T} dt \int d^3x \, \mathcal{L}_\theta \right)$$

$$\mathcal{V}_4 \equiv T/V_3, \qquad A_\mu(1/T, \mathbf{x}) = A_\mu(0, \mathbf{x}), \qquad E(\theta) = F(\theta, 0)$$

In pure gauge theory θ is a dimensionless RG invariant parameter, i.e. it does not renormalize in appropriate RG schemes, such as the $\overline{\mathrm{MS}}$ scheme

The ground-state/free energy can be parametrized as

(assuming analiticity at $\theta = 0$ (CP is not broken at $\theta = 0$, Vafa, Witten, 1984))

$$\mathcal{F}(\theta, T) \equiv F(\theta, T) - F(0, T) = \frac{1}{2}\chi(T)\theta^2 s(\theta, T)$$

$\chi(T) = \partial^2 F(\theta, T)/\partial\theta^2|_{\theta=0} = \int d^4x \langle q(x)q(0)\rangle_{\theta=0} = \langle Q^2\rangle_{\theta=0}/\mathcal{V}_4$ is the topological susceptibility, a measure of topological properties at $\theta = 0$

$s(\theta, T)$ is a dimensionless even function of θ such that $s(0, T) = 1$,
$\longrightarrow s(\theta, T) = 1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \cdots$

b_i are dimensionless RG invariant quantities, related to the zero-momentum correlation functions of $q(x)$, or the cumulants of $P(Q)$.
(V, Panagopoulos, PhysRep 2009)

e.g. $b_2 = -\chi_4/(12\chi)$ and $\chi_4 = \int d^4x_1 d^4x_2 d^4x_3 \langle q(0)q(x_1)q(x_2)q(x_3)\rangle_c|_{\theta=0}$

If $b_{2n} = 0$ then the distribution is Gaussian $P(Q) = \frac{1}{\sqrt{2\pi\langle Q^2\rangle}} \exp\left(-\frac{Q^2}{2\langle Q^2\rangle}\right)$

θ dependence of the spectrum

$\sigma(\theta) = \sigma\left(1 + s_2\theta^2 + ...\right)$ where σ is the string tension at $\theta = 0$.

$M(\theta) = M\left(1 + g_2\theta^2 + ...\right)$ where M is the 0^{++} glueball mass at $\theta = 0$

At $\theta \neq 0$, the lightest glueball state does not have a definite parity anymore, but it becomes a mixed state of 0^{++} and 0^{-+} glueballs.

The coefficients can be computed from appropriate correlators at $\theta = 0$, involving particle sources and $q(x)$

Within the large-N framework ($N \to \infty$, $g^2 N$ fixed)

the U(1)$_A$ problem is explained by assuming a nontrivial θ dependence at the leading (planar) $1/N$ order

$\longrightarrow \chi = \frac{f_s^2 m_s^2}{4N_f}$ or $\frac{4N_f}{f_\pi^2}\chi = m_{\eta'}^2 + m_\eta^2 - 2m_K^2$ (Witten, Veneziano, 1979)

Large-N scaling to $\mathcal{L}_\theta = \frac{1}{4}F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) - i\theta\frac{g^2}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x)$
$\longrightarrow$ the relevant scaling variable is $\bar{\theta} \equiv \theta/N$

$$f(\theta) \equiv \frac{F(\theta) - F(0)}{\sigma^2} = \frac{1}{2}C\theta^2(1 + b_2\theta^2 + b_4\theta^4 + ...) = N^2\bar{f}(\bar{\theta})$$

$\bar{f}(\bar{\theta})$ has a nontrivial large-N limit: $\frac{1}{2}C_\infty\bar{\theta}^2(1 + \bar{b}_2\bar{\theta}^2 + \bar{b}_4\bar{\theta}^4 + \cdots)$,
where $C \equiv \chi/\sigma^2 = C_\infty + c_2/N^2 + ...$, and $b_{2j} = \bar{b}_{2j}/N^{2j} + ...$

Multibranched $F(\theta)$, $F(\theta) - F(0) = \mathcal{A}\,\mathrm{Min}_k\,(\theta + 2\pi k)^2 + O\,(1/N)$ avoids the apparent incompatibility with periodicity in θ (Witten, AP 1980, PRL 1998).

This scenario is analytically verified within the theoretical laboratory of the 2D CP^{N-1} models with a N-component complex field z ($\bar{z}z = 1$)

$$\mathcal{L}_\theta = \frac{N}{2g}\overline{D_\mu z}\, D_\mu z - i\theta q(x), \qquad D_\mu = \partial_\mu + iA_\mu, \;\; A_\mu = i\bar{z}\partial_\mu z,$$

$$q(x) = \frac{1}{2\pi}\,\epsilon_{\mu\nu}\,\partial_\mu A_\nu, \qquad F(\theta) = -\frac{1}{V}\ln \int [dA] \exp\left(-\int d^2x \mathcal{L}_\theta\right)$$

They share several features with QCD: asymptotic freedom, topology, θ vacua.

Unlike 4D $SU(N)$ gauge theories, systematic $1/N$ expansion, keeping g fixed, around the large-N saddle-point solution.

Large-N scaling analogous to 4D SU(N) gauge theories at $T = 0$:

$$f(\theta) = \xi^2[F(\theta) - F(0)] = \frac{1}{2}(\chi\xi^2)\,\theta^2\left(1 + \sum_{n=1} b_{2n}\theta^{2n}\right) \approx N\bar{f}(\bar{\theta} \equiv \theta/N),$$

$$\bar{f}(\bar{\theta}) = \frac{1}{2}C_\infty\bar{\theta}^2(1 + \sum_{n=1}\bar{b}_{2n}\bar{\theta}^{2n}), \quad C_\infty = N(\chi\xi^2), \;\; \bar{b}_{2n} = N^{2n}b_{2n}$$

Analytic $1/N$ calculations confirm the large-N scaling

Semiclassically θ dependence arises from instantons.

Instantons: classical solutions which minimize the action within each topological sector, tunneling among n-vacua (Polyakov 1975, 't Hooft 1976)

$$A_I^\mu = 2\eta^{\mu\nu,a}\tau^a \frac{x_\nu}{x^2+\rho^2}, \qquad S(A_I) = 8\pi^2/g^2$$

Nontrivial θ dependence from the expansion of the functional integral in the sector with $Q = n$ about its minimum action.

$$Z = \int \mathcal{D}A e^{-S(A)+i\theta Q(A)} = \sum_n e^{i\theta n} \int \mathcal{D}A \delta(Q-n) e^{-S(A)}$$

$$\approx \sum_n e^{i\theta n} \int du \; e^{-S(A_I(u))} \mathrm{Det}[Q(A_I(u))]$$

The one-instanton contribution suggests that the θ dependence is exponentially small in N: $e^{-8\pi^2/g^2} e^{i\theta} = \left(e^{-8\pi^2/(g^2N)} e^{i\theta/N}\right)^N$

This conclusion is incorrect: the instanton gas approximation fails due to infrared divergences, large instantons are not suppressed

Dilute instanton-gas (DIG) approximation at finite T

At finite T, due to electric screening, only fields with integer Q can contribute to the functional integral: periodic instantons in $\beta = 1/T$
(Harrington, Shepard, PRD 1978; Gross, Pisarski, Yaffe, RMP 1981)

$$A_I^\mu = \Pi \bar{\eta}^{\mu\nu,a}(\tau^a/2i)\partial_\nu \Pi^{-1}, \qquad \Pi(t,\mathbf{x}) = 1 + \frac{(\pi\rho^2 T/r)\sinh(2\pi T r)}{\cos(2\pi T r) - \cosh(2\pi T t)}$$

T plays the role of infrared cutoff: $n_I(\rho) \sim e^{-S(A_I)} \sim e^{-[8\pi^2/g^2 + 2N(\pi\rho T)^2]}$

DIG approximation summing over n_+ instantons and n_- antiinstantons:

$$\begin{aligned} Z_\theta &= \mathrm{Tr}\, e^{-H_\theta/T} \approx \sum \frac{1}{n_+! n_-!} (\mathcal{V}_4 D)^{n_+ + n_-} e^{-\frac{8\pi^2(n_+ + n_-)}{g^2} + i\theta(n_+ - n_-)} \\ &= \exp\left[\cos\theta \times 2\mathcal{V}_4 D \times e^{-8\pi^2/g^2}\right] \end{aligned}$$

therefore $\mathcal{F}(\theta,T) \equiv F(\theta,T) - F(0,T) \approx \chi(T)\,(1 - \cos\theta)$

At high T ... dilute instanton-gas (DIG) approximation

At one loop $\partial F/\partial\theta = \sin\theta \int_0^\infty d\rho n_I(\rho) \sim \sin\theta \times T^4 e^{-8\pi^2/g^2(T)}$

$\mathcal{F}(\theta, T) \approx \chi(T)\,(1 - \cos\theta)\,, \qquad \chi(T) \approx T^4 \exp[-8\pi^2/g^2(T)] \sim T^{-\frac{11}{3}N+4},$

using $8\pi^2/g^2(T) \approx (11/3)N \ln(T/\Lambda) + O(\ln\ln T/\ln^2 T)$

DIG is a good approximation when the overlap between instantons becomes negligible, thus at large T where $\chi(T) \sim \rho_{\rm Inst}$ is suppressed

The high-T DIG θ dependence qualitatively differs from that at $T = 0$:

- analytic and periodic θ dependence
- The large-N scaling is not realized by the DIG approximation: the relevant variable for the instanton gas is θ, and **not** θ/N
- $\chi(T)$ gets exponentially suppressed in the large-N regime, suggesting a rapid decrease of the topological activity with increasing N at high T

• The low-T and high-T phases are separated by a $1^{\rm st}$-order deconfinement transition, at $T_c/\sqrt{\sigma} \approx 0.545(2) + 0.46(2)/N^2$ (Lucini, etal, 2004,2012) getting stronger with increasing N, $L_h \sim N^2$

• for $T \ll T_c$ → large-N scaling with θ/N as scaling variable → $\chi/\sigma^2 \approx C_\infty + c/N^2$ and $b_k \approx \bar{b}_k/N^k$.

Does it extend up to T_c^-?

• for $T \gg T_c$ → analytic θ dependence by DIG approximation: $\mathcal{F} \approx \chi(T)(1 - \cos\theta)$ with $\chi(T) \sim T^{-\frac{11}{3}N+4}$.

Does it extend down to T_c^+?

⟶ The low-T and high-T θ dependence changes around the deconfinement transition ???

Some hints also from models like ADS-CFT, holographic models, etc... (Witten, PRL 1998; Parnashev, Zhitnisky PRD 1998; Unsal PRD 2012, etc)

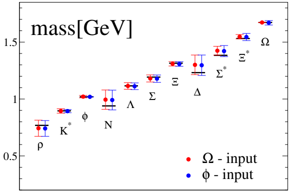
A quantitative study of θ dependence requires a nonpertubative approach: Wilson lattice formulation of QCD, from the critical continuum limit of a statistical 4D lattice model: $Z = \int [dU] \exp(-S_L)$

$$S_L = -\frac{2a^4}{g_0^2} \sum \mathrm{ReTr}\,[U_\mu(x) U_\nu(x+a\hat{\mu}) U_\mu^\dagger(x+a\hat{\nu}) U_\nu^\dagger(x)], \quad U_\mu \in SU(N)$$

Formally, in the $a \to 0$ limit, one recovers $S = \int d^4x \frac{1}{2g_0^2} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu}$.

The statistical theory develops a mass gap, and therefore a length scale ξ. The continuum theory is defined in the critical limit $g_0^2 \to 0$, when $\xi \to \infty$: $\xi = a \exp \int^{g_0} dg/\beta_L(g) \sim a(b_0 g_0^2)^{-b_1/2b_0^2} \exp[1/(2b_0 g_0^2)]$, where $\beta_L = a dg_0/da = -b_0 g_0^3 - b_1 g_0^5 + \ldots$

The lattice formulation lends itself to statistical-physics techniques, such as MC simulations



Topology from the lattice is a nontrivial issue: The topology on the lattice is strictly trivial, because its configuration space is simply connected. The topological properties are expected to be recovered in the continuum limit.

From a QFT point view, problems are related to the peculiar singular behavior of the two-point function $G(x) = \langle q(0)q(x)\rangle$ when $x \to 0$ ($G(x) < 0$ for $|x| \neq 0$, while $\int dx\, G(x) = \langle Q^2\rangle/V > 0$, and $G(x) \sim r^{-8}(\ln r)^{-2}$)

Various methods to estimate Q:

• Smoothing methods read Q after smoothing the configurations.
Several variants, such as cooling, smearing, gradient flow, ..., equivalent in the relevant scaling region $\beta \gtrsim 6$ for SU(3).

• The fermionic definition through the index of the overlap Dirac operator provides a well–defined estimator for Q, but at a much higher computational cost.

• Other methods: Geometrical, Off-equilibrium, etc...

Several numerical checks and comparisons have shown that they provide accurate and reliable results.

The complex nature of the θ term $i\theta q(x)$ in the Euclidean QCD Lagrangian prohibits a direct MC simulation at $\theta \neq 0$.

Information on the θ dependence of physically relevant quantities can be obtained by computing their expansion around $\theta = 0$:

Expansion around $\theta = 0$: $F(\theta) - F(0) = \frac{1}{2}\chi\theta^2(1 + b_2\theta^2 + b_4\theta^4 + \ldots)$

χ and b_{2n} from correlation functions $\langle q(x_1)q(x_2)\ldots q(x_{2n})\rangle$ at $\theta = 0$,

$$b_2 = -\frac{\chi_4}{12\chi}, \qquad \chi_4 = \frac{1}{V}\left[\langle Q^4\rangle - 3\langle Q^2\rangle^2\right]_{\theta=0}, \qquad Q = \sum_x q(x)$$

$$b_4 = \frac{\chi_6}{360\chi}, \qquad \chi_6 = \frac{1}{V}\left[\langle Q^6\rangle - 15\langle Q^2\rangle\langle Q^4\rangle + 30\langle Q^2\rangle^3\right]_{\theta=0}$$

In the continuum limit, $b_{2k,L} \approx b_{2k} + a^2\sigma^2$ for $a \to 0$

$b_{2n} \to$ deviations from a Gaussian $P(Q) = \frac{1}{\sqrt{2\pi\langle Q^2\rangle}} \exp\left(-\frac{Q^2}{2\langle Q^2\rangle}\right)$

They require large statistics, due to the cancellation of volume factors

Alternatively, imaginary θ term $\theta_i = -i\theta$,

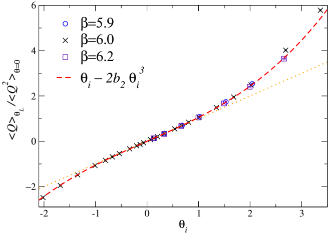
$$Z_L = \int [dA] \exp\left(-S_L + \Theta_L Q_L\right), \quad S_L = -\frac{2}{g_0^2} \sum_{x,\mu>\nu} \mathrm{Re}\,\mathrm{Tr}\, \Pi_{\mu\nu}(x),$$

$Q_L \equiv \sum_x q_L(x)$ is a discretization of $q(x)$, $\theta_i = Z_\theta \Theta_L$ where $Z_\theta = \langle Q Q_L \rangle / \langle Q^2 \rangle|_{\theta=0}$

Replacing $\theta \equiv -i\theta_i$ in the free energy, to be eventually extended to real θ

$$\Phi(\theta_i) \equiv \mathcal{F}(-i\theta_i) = -\frac{1}{2}\chi\theta_i^2 s(-i\theta_i) = -\frac{1}{2}\chi\theta_i^2 \left(1 - b_2\theta_i^2 + b_4\theta_i^4 + \cdots\right)$$

- Scaling for $|\theta_i| \lesssim \pi$ for $N = 3$, thus a continuum limit for any value of θ_i. (Panagopoulos, V, JHEP 2011)
- The θ_i dep is well described by the first few terms of the expansion around $\theta = 0$.
- θ-dep of T_c (D'Elia, Negro, PRL 2012)

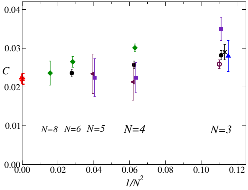


θ dependence at $T = 0$

• $\chi \equiv \partial^2 F(\theta)/\partial\theta^2|_{\theta=0} \neq 0$ for SU(3): $\chi/\sigma^2 = 0.028(2)$ by various methods in agreement with the WV relation $\frac{4N_f}{f_\pi^2}\chi = m_{\eta'}^2 + m_\eta^2 - 2m_K^2$

• Nonzero large-N limit: $\chi/\sigma^2 = 0.022(2)$, from MC simulations for $N > 3$ (by Cundy, Del Debbio, Lucini, Panagopoulos, Teper, V., Wenger, ...)

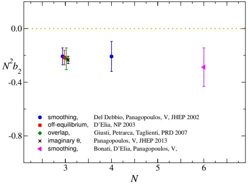
They support the expected large-N behavior: $\chi/\sigma^2 = C_\infty + c_2/N^2$



• Nonzero higher-order terms of $F(\theta) - F(0) = \frac{1}{2}\chi\theta^2(1 + b_2\theta^2 + b_4\theta^4 + ...)$ SU(3) estimates: $b_2 = -0.026(3)$ and $|b_4| \lesssim 0.001$ (using various methods to determine Q)

• Vanishing large-N limit of $b_k = O(N^{-k})$, results consistent with $b_2 \approx \bar{b}_2/N^2$, $\bar{b}_2 \approx -0.2$,

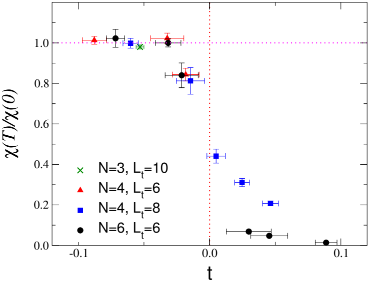
$\longrightarrow$ Deviations from a simple Gaussian behavior are already small at $N = 3$ $\longrightarrow$ $N = 3$ is already large!



- $\chi = \partial^2 F(\theta, T)/\partial\theta^2|_{\theta=0}$ at finite T

$\chi(T)/\chi(T=0)$ vs $t \equiv T/T_c - 1$
across the transition $\longrightarrow$

Several MC results (Alles, Bonati, Del Debbio, D'Elia, Di Giacomo, Lucini, Panagopoulos, Teper, V., Wenger, ...)



- χ remains substantially unchanged in the low-T confined phase.
- Sharp change across the first-order transition, likely discontinuous
- In the high-T phase χ is suppressed, especially at large N

in qualitative agreement with one-loop DIG $\chi(T) \sim T^{-\frac{11}{3}N+4}$ for $T \gg T_c$

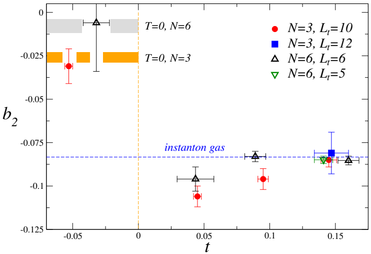
but larger T is necessary for a quantitative check of the one-loop DIG approximation, due to logarithmic corrections

• Higher-order terms of $F(\theta,T) = \frac{1}{2}\chi\theta^2(1 + b_2\theta^2 + b_4\theta^4 + \cdots)$ provide a more significant probe of DIG regimes, avoiding the problem of the logarithmic corrections of the prefactor

High-stat MC for $N = 3, 6$ to check large N (smoothing techniques for Q)

(Bonati, D'Elia, Panagopoulos, V, PRL 2013)

b_{2k} are compared with $T = 0$ results and DIG approx ($b_2 = -1/12$) $\longrightarrow$



• Sharp change across the deconfinement transition, likely discontinuous

• For $T > T_c$, rapid approach to DIG θ dependence, with deviations visible only for $t \approx 0.05$. The approach appears faster with increasing N.

• $b_4 = 0.0024(4)$ for $N = 6$ and $t = 0.09$ to be compared with $b_4 = 1/360 = 0.00277$

Deviations from dilute instanton gas at $t \equiv (T - T_c)/T_c \lesssim 0.1$

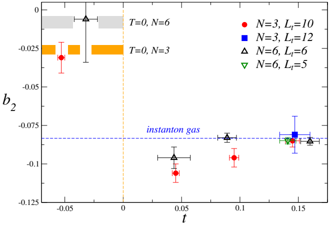
parametrized by a virial-like expansion: adding a correction proportional to the square of the instanton density

Since $\chi(T) \sim \rho_{\text{inst}}$,

$\mathcal{F}(\theta, T) \approx \chi(1 - \cos\theta) + \chi^2 \kappa(\theta) + O(\chi^3)$

$\kappa(\theta) = \sum_{k=2} c_{2k} \sin(\theta/2)^{2k}$

Thus, $b_2 = -\frac{1}{12} + \frac{1}{8} c_4 \chi + O(\chi^2)$



$\chi \sim T^{-11N/3+4}$ implies a rapid approach to the asymptotic DIG value, which becomes faster with increasing N.

The hard-core approximation of instanton interactions give a negative correction, i.e. $c_4 < 0$, explaining the approach from below to the DIG value $b_2 = -1/12$.

Summary of the θ dependence in 4D SU(N) gauge theories

$$\mathcal{F}(\theta,T) \equiv F(\theta,T) - F(0,T) = \frac{1}{2}\chi(T)\theta^2 \left(1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \cdots\right)$$

• Low-T phase characterized by a large-N scaling with θ/N as relevant variable: $\chi/\sigma^2 \approx C_\infty + c/N^2$ and $b_k \approx \bar{b}_k/N^k$

• Sharp change across the deconfinement transition, likely discontinuous, sharper with increasing N

• High-T phase: large-N scaling is lost, the topological activity is much reduced. The dilute instanton-gas regime sets in just above T_c, giving an analytic dependence $F(\theta) - F(0) \approx \chi(T)(1 - \cos\theta)$

• MC simulations of lattice theory support the above scenario

QCD: $\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \sum_f \bar{\psi}_f D\psi_f + \sum_f \bar{\psi}_f(\mathrm{Re}\, m_f + i\mathrm{Im}\, m_f\, \gamma_5)\psi_f - i\theta q$

- θ and $\mathrm{Im}\, m_f$ are related by chiral transformations $\psi \to e^{i\alpha\gamma_5}\psi$

- no θ-dep if $m_f = 0$, but $m_u \approx m_d/2 > 0$.

- θ is not RG invariant, indeed in the quark massless limit

$$\begin{pmatrix} i2N_f\, q(x) \\ \partial_\mu j^5_\mu(x) \end{pmatrix}_R = \begin{pmatrix} 1 & z-1 \\ 0 & z \end{pmatrix} \begin{pmatrix} i2N_f\, q(x) \\ \partial_\mu j^5_\mu(x) \end{pmatrix}_B$$

where $z = 1 + \frac{g^4}{16\pi^4}\frac{3c_F}{8}N_f\frac{1}{\epsilon} + O\left(g^6\right)$, and $\epsilon = 2 - d/2$, $c_F = (N^2-1)/(2N)$

- However, the residual θ parameter when $\mathrm{Im}\, m_f = 0$ does not renormalize, essentially due to the nonrenormalizability of the anomaly equation $\partial_\mu j^5_\mu(x) = i2p(x) + i2N_f q(x)$

- The residual θ is a physical parameter, $|\theta| \lesssim 10^{-9}$ from experimental bounds of NEDM.

• Large-N scaling at $T = 0$ solving U$(1)_A$ problem (Witten, Veneziano, 1979)

• Finite-T transition from hadrons to quark-gluon plasma (crossover at the physical mass values)

• At high-T the dilute-instanton gas scenario applies to full QCD as well

Analogous suppression of θ-dep at high-T $\longrightarrow$ suppression of the U$(1)_A$ breaking in the quark-gluon plasma.

• Crossover between the two regimes likely around the transition

• Is the U$(1)_A$ symmetry restored at finite T?

DIG suggests that the U$(1)_A$ symmetry is suppressed but not exactly recovered at finite T, due to residual small instanton effects in the chiral limit.

In two-flavor QCD, DIG predicts $\chi \sim m_f^2 \, T^{-11N/3+16/3}$, but also a residual $\mathrm{U}(1)_A$ breaking from Dirac zero modes

$$\chi_\pi - \chi_\delta \sim T^{-11N/3+16/3}$$

which are the susceptibilities of the π ($\bar\psi\vec\tau\gamma_5\psi$) and δ ($\bar\psi\vec\tau\psi$) channels

The suppression of the θ dependence and U$(1)_A$ breaking may be relevant for the nature of the finite-T transition in the chiral limit, in particular for two light flavors

In the case of a continuous transition its suppression would lead to a change of the universality class of the two-light-flavor chiral transition

$$[U(2)_L \otimes U(2)_R] \,/\, U(2)_V \;\neq\; [SU(2)_L \otimes SU(2)_R] \,/\, SU(2)_V \;=\; O(4)/O(3)$$

(Pelissetto, V, PRD 2013, Pisarski, Wilczek, PRD 1984)

$\longrightarrow$ Numerical studies of full QCD show the suppression (apparently not complete) of the $U(1)_A$-breaking effects at, and above, the finite-T transition.

Hints for an early onset of DIG around/above the hadronic-quarkgluon (almost) transition by looking at the behavior of the susceptibilities of different channels (Bazavov, etal, PRD 2012; Buchoff etal, 2013, Cossu et al, 2013), analogously to pure SU(N) gauge theory