# Covariant phase space and multisymplectic geometry

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# The covariant phase

Main principle:

The set of solutions $\mathcal{E}$ of a Euler–Lagrange is endowed with a *God given* canonical pre-symplectic structure

In Mechanics:

The Cauchy data at two different times provides two charts on $\mathcal{E}$ which carries the same symplectic structure, because the flow is Hamiltonian (Lagrange, Hamilton, Jacobi, Poincaré, Cartan, Dedecker, Souriau,...).

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- general, synthetic and geometric presentation by J. Kijiowski–W. Szczyrba in 1976, by using the multisymplectic formalism;
- rediscovered by C. Crnkovic–E. Witten in and G. Zuckerman in 1987, being inspired by the variational complex of F. Takens (1979), A. Vinogradov (1984);

# The multisymplectic formalism

... in a few words.

Lagrangian formulation of dynamics:

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  where dvol is a volume $n$-form on $\mathrm{X}^n$
- the Euler–Lagrange system of equations is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^i_\mu}(x, \mathbf{u}(x), d\mathbf{u}_x)\right) = \frac{\partial L}{\partial y^i}(x, \mathbf{u}(x), d\mathbf{u}_x). \qquad (1)$$

# Legendre hypothesis

Set

$$p_i^\mu := \frac{\partial L}{\partial v_\mu^i}(x^\mu, y^i, v_\mu^i)$$

and assume that $(x^\mu, y^i, v_\mu^i) \longmapsto (x^\mu, y^i, p_i^\mu)$ is a good change of variable.

- consider the Hamiltonian function:

$$H(x^\mu, y^i, p_i^\mu) := p_i^\mu v_\mu^i - L(x^\mu, y^i, v_\mu^i)$$

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- then the Euler–Lagrange equations reads

$$\left\{ \begin{array}{rcl} \dfrac{\partial \mathbf{u}^i}{\partial x^\mu}(x) & = & \dfrac{\partial H}{\partial p_i^\mu}(x^\mu, \mathbf{u}^i(x), \mathbf{p}_i^\mu(x)) \\ \dfrac{\partial \mathbf{p}_i^\mu}{\partial x^\mu}(x) & = & -\dfrac{\partial H}{\partial y^i}(x^\mu, \mathbf{u}^i(x), \mathbf{p}_i^\mu(x)), \end{array} \right. \quad (2)$$

the so-called **De Donder–Weyl** system of equations.

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- Each of these two filiations were motivated by different questions.
- a geometrization of these ideas was initiated by P. Dedecker (1953)
- and fully accomplished by the Polish school starting with an (unpublished) seminar in Warsaw by W. Tulczyjew after 1965 and followed by publications by J. Kijowski, K. Gawedski, W. Sczcyrba in the seventies.

# Geometrization

In $\mathrm{X}^n \times \mathrm{Y}^k \times \mathrm{End}(\mathrm{X}^n, \mathrm{Y}^k)^* \ni (x^\mu, y^i, p_i^\mu)$ consider the $n$-dimensional submanifold $\Gamma$ defined by:

$$y^i = \mathbf{u}^i(x) \quad \text{and} \quad p_i^\mu = \frac{\partial L}{\partial v_\mu^i}(x^\mu, \mathbf{u}^i(x), \partial_\mu \mathbf{u}^i(x)).$$

and the pre-**multisymplectic** $(n+1)$-form

$$\omega := \mathrm{d}p_i^\mu \wedge \mathrm{d}y^i \wedge \mathrm{dvol}(\partial_\mu, \cdots) - \mathrm{d}(H(x,y,p)) \wedge \mathrm{dvol}.$$

Then the Hamilton-Volterra–De Donder–Weyl system of equations reads:

$$\forall \xi \text{ (vector field)}, \quad \omega(\xi, \cdots)|_\Gamma = 0.$$

We can replace maps $\mathbf{u} : \mathrm{X}^n \longrightarrow \mathrm{Y}^k$ by sections $\mathbf{u} : \mathcal{X}^n \longrightarrow \mathcal{Z}^{n+k}$ of a vector bundle

$$\pi_{\mathcal{X}} : \mathcal{Z}^{n+k} \longrightarrow \mathcal{X}^n.$$

Consider the vector bundle $\Lambda^n T^*\mathcal{Z}^{n+k}$ of $n$-forms over $\mathcal{Z}^{n+k}$.
There is a canonical $n$-form $\theta$ on $\Lambda^n T^*\mathcal{Z}^{n+k}$ defined by:
$\forall z \in \mathcal{Z}^{n+k}$, $\forall p \in \Lambda^n T^*_z\mathcal{Z}^{n+k}$,

$$\theta_{(z,p)}(X_1, \cdots, X_n) := p(\mathrm{d}\pi_{\mathcal{X}}(X_1), \cdots, \mathrm{d}\pi_{\mathcal{X}}(X_n)).$$

Then

$$\omega := \mathrm{d}\theta$$

is a **multisymplectic** $(n+1)$-form.

# Geometric generalizations 2

The multisymplectic manifold $(\Lambda^n T^*\mathcal{Z}^{n+k}, \omega)$ is too large. We restrict ourself to a submanifold of $\Lambda^n T^*\mathcal{Z}^{n+k}$.

- Call a tangent vector $v \in T_z\mathcal{Z}$ s.t. $\mathrm{d}\pi_{\mathcal{X}}(v) = 0$ a **vertical** vector field;

This gives us the right generalization of:

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- define $\Lambda^n_2 T^*\mathcal{Z}$ to be the set of $(z, p) \in \Lambda^n T^*\mathcal{Z}$ s.t.

$$\text{if } v \text{ and } w \text{ are vertical, then } p(v, w, \cdots) = 0;$$

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- set $\theta_2$ : the restriction of $\theta$ to $\Lambda^n_2 T^*\mathcal{Z}$.
- (optional) if $\mathcal{H} : \Lambda^n_2 T^*\mathcal{Z} \longrightarrow \mathbb{R}$ is a Hamiltonian function, restrict further to the submanifold $\mathcal{H} = 0$ (pre-$n$-plectic).

This gives us the right generalization of:

$$\omega := dp_i^\mu \wedge dy^i \wedge d\text{vol}(\partial_\mu, \cdots) - d(H(x, y, p)) \wedge d\text{vol}.$$

# Multisymplectic manifolds : general framework

An *n-plectic manifold* $\mathcal{N}$ is a manifold equipped with an $(n+1)$-form $\omega$ s.t.

- $\omega$ is closed: $d\omega = 0$;

A Hamiltonian function on an $n$-plectic manifold $(\mathcal{N}, \omega)$ is a function $\mathcal{H} : \mathcal{N} \longrightarrow \mathbb{R}$ s.t. $d\mathcal{H} \neq 0$.

An $n$-dimensional submanifold $\Gamma \subset \mathcal{N}$ is a solution of the generalized Hamilton equations if:

$$\text{for any vector field } \xi, \quad d\mathcal{H}(\xi) = 0 \Longrightarrow \omega(\xi, \cdots)|_\Gamma = 0$$

and $\Gamma$ is then called a Hamiltonian $n$-curve.

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- $\omega$ is closed: $d\omega = 0$;
- $\omega$ is non degenerate: for any vector $\xi$, if $\omega(\xi, \cdots) = 0$, then $\xi = 0$.

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# An example

The scalar fields $\mathbf{u} : \mathbb{R}^n \longrightarrow \mathbb{R}$ which are solutions of the Klein–Gordon equation

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \Delta \mathbf{u} + V'(\mathbf{u}) = 0.$$

On $\mathcal{N} := \{(x^\mu, y, p^\mu, e) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}\}$ set

$$\omega = \mathrm{d}e \wedge \beta + \mathrm{d}p^\mu \wedge \mathrm{d}y \wedge \beta_\mu,$$

where $\beta := \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}$ and $\beta_\mu := \beta(\partial_\mu, \cdots)$; and

$$\mathcal{H}(x^\mu, y, p^\mu, e) := e + \frac{1}{2}\eta_{\mu\nu} p^\mu p^\nu + V(y);$$

Then $\mathbf{u}$ is a solution of the Klein–Gordon equation iff the submanifold $\Gamma$ of equation $y = \mathbf{u}(x)$, $\mathbf{p}^\mu(x) = \eta^{\mu\nu}\partial_\nu \mathbf{u}(x)$ and $\mathcal{H}(x^\mu, \mathbf{u}(x), \mathbf{p}^\mu(x), \mathbf{e}(x)) = 0$ is a solution of:

for any vector field $\xi$, $\quad \mathrm{d}\mathcal{H}(\xi) = 0 \Longrightarrow \omega(\xi, \cdots)|_\Gamma = 0.$

# Pre-multisymplectic manifolds : general framework

A pre-$n$-plectic manifold $\mathcal{N}$ is a manifold equipped with an $(n+1)$-form $\omega$ and a 'volume' $n$-form $\beta$ s.t. $\omega$ is closed: $d\omega = 0$ and $\beta$ does not vanish.

Fundamental example: let $(\mathcal{N}, \omega)$ be an $n$-plectic manifold and $\mathcal{H} : \mathcal{N} \longrightarrow \mathbb{R}$ is a Hamiltonian function. Then the submanifold

$$\mathcal{N}^0 := \mathcal{H}^{-1}(0) := \{z \in \mathcal{N} |\ \mathcal{H}(z) = 0\}$$

with the pre-$(n+1)$-plectic form $\omega|_{\mathcal{N}^0}$ and the 'volume' $n$-form $\beta = \omega(\tau, \cdots)|_{\mathcal{N}^0}$ (where $\tau$ is a vector field s.t. $d\mathcal{H}(\tau) = 1$) is a pre-$n$-plectic manifold.

Then the dynamical equations read

$$\forall \xi, \quad \omega(\xi, \cdots)|_\Gamma = 0 \quad \text{and} \quad \beta|_\Gamma \neq 0.$$

and $\Gamma$ is then called a Hamiltonian $n$-curve.

# Another example of a pre-multisymplectic manifold

The Palatini action for gravity (Hélein–Kouneiher, Rovelli, 2004). Let $\mathcal{X}$ be a 4-dimensional manifold, $\mathcal{V}$ a rank 4 vector bundle over $\mathcal{X}$ equipped with a Minkowski metric. Consider the manifold $\mathcal{N}$ which is the configuration space of pairs $(\alpha, \nabla)$, where $\nabla$ is a connection on $\mathcal{V}$ which respects the metric and $\alpha$ is a 1-form on $\mathcal{X}$ with values in $\mathcal{V}$. Given a local trivialization $\mathcal{V} \simeq \mathcal{X} \times \mathbb{R}^4$, $\mathcal{N}$ can be identified with 1-forms $(a^I, A^I_J)$ on $\mathcal{X}$ with coefficients in $\mathbb{R}^4 \times \mathfrak{so}(1,3)$.

On $\mathcal{N}$ we define the 4-form

$$\theta := 1/4!\ \epsilon_{IJKL}\eta^{LN} a^I \wedge a^J \wedge F^K_N ,$$

where $F^I_J := dA^I_J + A^I_K \wedge A^K_J$. Then $\mathcal{N}$ with the pre-4-plectic 5-form $\omega := \mathrm{d}\theta$ and the volume 4-form $\beta := a^0 \wedge a^1 \wedge a^2 \wedge a^3$ is a pre-4-plectic manifold.

The dynamical equations $\omega(\xi, \cdots)|_\Gamma = 0$, $\forall \xi$ and $\beta|_\Gamma \neq 0$ are the (Palatini) Einstein equations for gravity.

## Variational principle

Any *n*-dimensional submanifold Γ on which $\beta$ does not vanish is a critical point of

$$\mathcal{A}[\Gamma] := \int_\Gamma \theta$$

iff it is a Hamiltonian *n*-curve.

# Basic results in a pre-multisymplectic manifold $\mathcal{N}$, 2

## Observable $(n-1)$-forms

An $(n-1)$-form $F$ on $\mathcal{N}$ is observable if there exists a vector field $\xi_F$ on $\mathcal{N}$ s.t.

$$\mathrm{d}F + \omega(\xi_F, \cdots) = 0.$$

## Brackets

Given two observable $(n-1)$-forms $F$ and $G$ their bracket is the observable $(n-1)$-form defined by

$$\{F, G\} := \omega(\xi_F, \xi_G, \cdots) = \mathrm{d}G(\xi_F, \cdots) = -\mathrm{d}F(\xi_G, \cdots).$$

Then $\{F, G\} + \{G, F\} = 0$ and

$$\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = \mathrm{d}(\omega(\xi_F, \xi_G, \xi_G, \cdots)).$$

## Observable functionals

Given a (codimension 1) hypersurface $\Sigma$ in $\mathcal{N}$ and an observable $(n-1)$-form $F$, one defines the functional on the set of $n$-dimensional submanifolds $\Gamma$

$$\begin{array}{rccc} \displaystyle\int_\Sigma F: & \{\Gamma\} & \longrightarrow & \mathbb{R} \\ & \Gamma & \longmapsto & \displaystyle\int_{\Sigma\cap\Gamma} F. \end{array}$$

If $\Gamma$ is a Hamiltonian $n$-curve, then this functional depends only on the homology class of $\Sigma$. We also define the Poisson bracket

$$\left\{\int_\Sigma F, \int_\Sigma G\right\} := \int_\Sigma \{F, G\}.$$

This bracket coincides with the standard bracket.

# Back to the covariant phase approach

Question:

How to read and compute the canonical pre-symplectic structure on the set of solutions $\mathcal{E}$ of a Euler–Lagrange ?

Several answers:

- Peierls brackets ?

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- multisymplectic approach...

# A short presentation using multisymplectic formalism

Recall that $\mathcal{E}$ is the set of Hamiltonian $n$-curves $\Gamma$ for some $\mathcal{H}$. Assume that $\omega = d\theta$, for some $n$-form $\theta$. For any $\Gamma \in \mathcal{E}$,

$$\text{an infinitesimal deformation of } \Gamma \quad \longleftrightarrow \quad \text{a vector } \delta\Gamma \in T_\Gamma\mathcal{E}$$
$$\longleftrightarrow \quad \text{a Jacobi vector field } \xi \text{ along } \Gamma$$

where a Jacobi vector field $\xi$ is a tangent vector field along $\Gamma$ which is a solution of:

$$\forall \zeta, \quad (L_\xi \omega)(\zeta, \cdots)|_\Gamma = 0.$$

We then write

$$\delta\Gamma = \int_\Gamma \xi \quad \text{and} \quad \Theta_\Gamma^\Sigma(\delta\Gamma) := \int_{\Gamma\cap\Sigma} \theta(\xi, \cdots).$$

## A general formula

Assume that $\Gamma$ is an $n$-dimensional submanifold and that $\Sigma_1$ and $\Sigma_2$ are two 'slices' (hypersurfaces) in the same homology class. Let $\mathcal{D}_1^2$ be the portion of $\mathcal{N}$ between $\Sigma_1$ and $\Sigma_2$, i.e.

$$\partial\mathcal{D}_1^2 = \Sigma_2 - \Sigma_1.$$

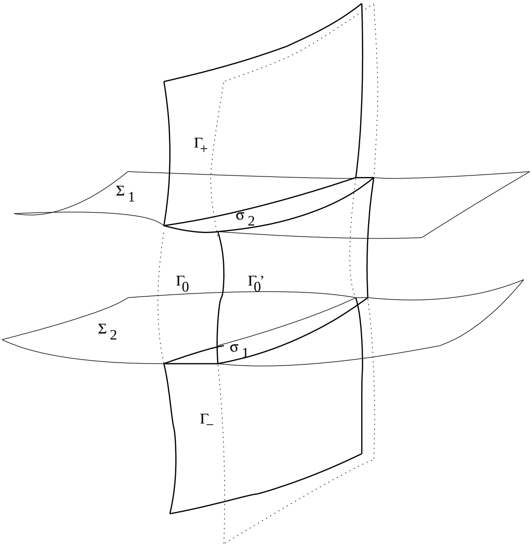
Then, if $\delta\Gamma = \int_\Gamma \xi$,

$$\delta(S_{\Sigma_1}^{\Sigma_2})_\Gamma(\delta\Gamma) - \Theta_\Gamma^{\Sigma_2}(\delta\Gamma) + \Theta_\Gamma^{\Sigma_1}(\delta\Gamma) = \int_{\Gamma\cap\mathcal{D}_1^2} \omega(\xi, \cdots)$$

where

$$S_{\Sigma_1}^{\Sigma_2}(\Gamma) := \int_{\Gamma\cap\mathcal{D}_1^2} \theta = \text{action between } \Sigma_1 \text{ and } \Sigma_2.$$

# A general formula: picture



# First consequence of the general formula

The right hand term

$$\int_{\Gamma\cap\mathcal{D}_1^2} \omega(\xi,\cdots)$$

vanishes iff $\Gamma$ is a Hamiltonian $n$-curve. If so

$$\Theta^{\Sigma_2} - \Theta^{\Sigma_1} = \delta(S_{\Sigma_1}^{\Sigma_2})$$

is exact on $\mathcal{E}$ and so its differential $\Omega := \delta\Theta^{\Sigma}$ does not depend on $\Sigma$ and is hence a 2-form on $\mathcal{E}$: the canonical pre-symplectic form.

# Second consequence of the general formula

Conversely if the left hand side

$$\delta(S_{\Sigma_1}^{\Sigma_2})_\Gamma(\delta\Gamma) - \Theta_\Gamma^{\Sigma_2}(\delta\Gamma) + \Theta_\Gamma^{\Sigma_1}(\delta\Gamma)$$

vanishes for all $\Sigma_2$ and $\Sigma_1$ (and for 'all' $\xi$'s), then $\Gamma$ is a solution of the Hamilton equations: this is a generalization (see De Donder) of the Cartan principle of dynamics by using the invariant integrals of Poincaré and Cartan.

# Conclusion

There are three main formulations of the classical dynamics of fields and dynamics:

- the Lagrangian one;
- the generalized Hamilton equations (multisymplectic version);
- the Poincaré–Cartan integral invariants.

The multisymplectic formalism provides an easy way to compare these formulations.

The secondary calculus of A. Vinogradov is an alternative way, tehcnically more complex, but more precise.

# Perspectives

Some direction to develop along these points of view:

- the search for expressions for first integrals in terms of (convergent) series (F. H.– R.D. Harrivel, arXiv:0704.2674);
- exploring the BRST and the Batalin–Vilkovisky theory.

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