Anomalies, gauge field topology, and the lattice

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Three sources of chiral symmetry breaking in QCD

- spontaneous breaking $\langle\overline{\psi}\psi\rangle \neq 0$
  - explains lightness of pions
- implicit breaking of $U(1)$ by the anomaly
  - explains why η' is not so light
- explicit breaking from quark masses
  - pions are not exactly massless

Rich physics from the interplay of these three effects

Based on very old ideas

- Dashen, 1971: possible spontaneous strong CP violation
  - before QCD!
- 't Hooft, 1976: ties between anomaly and gauge field topology
- Fujikawa, 1979: fermion measure and the anomaly
- Witten, 1980: connections with effective Lagrangians
  - many extensions shortly after
  - I won't use the large N aspects here
- MC 1995: Why is chiral symmetry so hard on the lattice
  - Z_n symmetry and transition at $\Theta = \pi$

Why rehash old ideas?

- consequences have raised bitter recent controversies

Axial anomaly in N_f flavor massless QCD

- leaves behind a residual Z_{N_f} flavor-singlet chiral symmetry
- tied to gauge field topology and the QCD theta parameter

Consequences

- degenerate quarks with $m \neq 0$:
  - first order transition at $\Theta = \pi$
- sign of mass relevant for odd N_f:
  - perturbation theory incomplete
- $N_f = 1$: no symmetry for mass protection
- nontrivial N_f dependence: invalidates rooting

} controversial

Consider QCD with N_f light quarks and assume

- the field theory exists and confines
- spontaneous chiral symmetry breaking $\langle\overline{\psi}\psi\rangle \neq 0$
- $SU(N_f) \times SU(N_f)$ chiral perturbation theory makes sense
- anomaly gives η' a mass
- N_f small enough to avoid any conformal phase

Use continuum language

- imagine some non-perturbative regulator in place (lattice?)
  - momentum space cutoff much larger than Λ_{QCD}
  - lattice spacing a much smaller than $1/\Lambda_{QCD}$

Construct effective potential V for meson fields

- V represents vacuum energy density for a given field expectation
- formally via a Legendre transformation

For simplicity initially consider

- degenerate quarks with small mass m
- N_f even
  - interesting subtleties for odd N_f

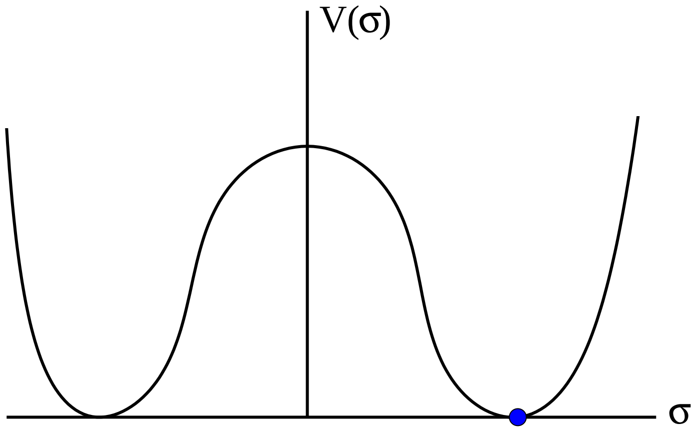
Work with composite fields

- $\sigma \sim \overline{\psi}\psi$
- $\pi_\alpha \sim i\overline{\psi}\lambda_\alpha\gamma_5\psi$
- $\eta' \sim i\overline{\psi}\gamma_5\psi$

λ_α: Gell-Mann matrices for $SU(N_f)$

Spontaneous symmetry breaking at $m = 0$

- $V(\sigma)$ has a double well structure
  - vacuum has $\langle\sigma\rangle = v \neq 0$
  - minimum of $V(\sigma) = \pm v$

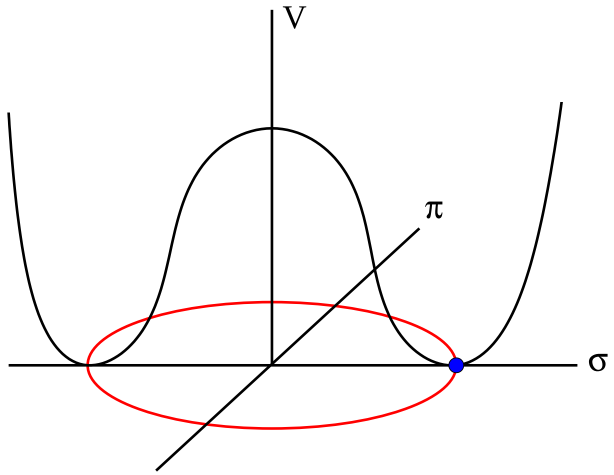


Ignore convexity issues

- phase separation occurs when a field is in a concave region

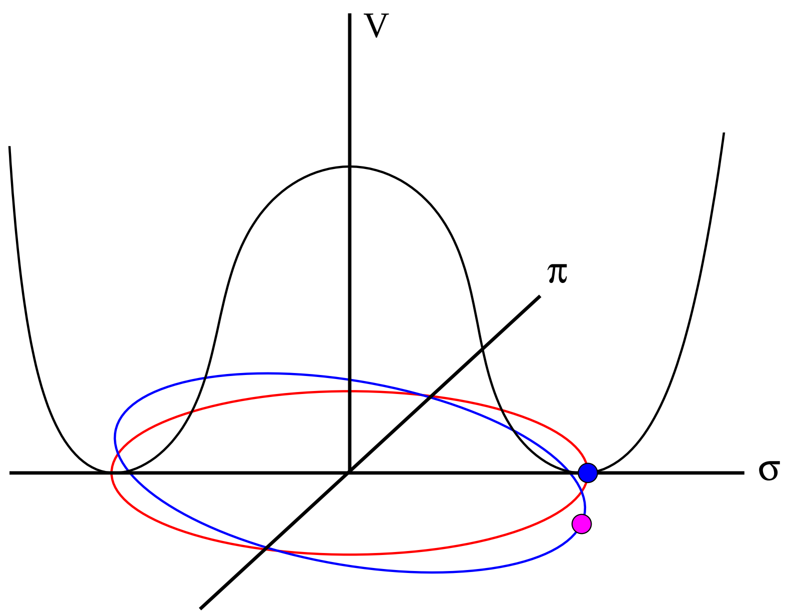
Nonsinglet pseudoscalars are Goldstone bosons

- symmetry under flavored rotations
  - $$\sigma \longrightarrow \cos(\phi)\sigma + \sin(\phi)\pi^\alpha$$ $$\pi^\alpha \longrightarrow \cos(\phi)\pi^\alpha - \sin(\phi)\sigma$$ $(N_f = 2)$
  - $\psi \longrightarrow e^{i\phi\gamma_5\lambda^\alpha}\psi$
- potential has $N_f^2 - 1$ "flat" directions
  - one for each generator of $SU(N_f)$



Small mass selects vacuum

- $V \to V - m\sigma$
- $\langle\sigma\rangle \sim +v \qquad \langle\pi\rangle = 0$
- Goldstones acquire mass $\sim \sqrt{m}$



Anomaly gives the η' a mass even if $m_q = 0$

- $m_{\eta'} = O(\Lambda_{QCD})$
- $V(\sigma, \eta')$ not symmetric under
  - $\psi \longrightarrow e^{i\phi\gamma_5}\psi$
  - $\sigma \longrightarrow \sigma\cos(\phi) + \eta'\sin(\phi)$
  - $\eta' \longrightarrow -\sigma\sin(\phi) + \eta'\cos(\phi)$

Expand the effective potential near the vacuum state $\sigma \sim v$ and $\eta' \sim 0$

- $V(\sigma, \eta') = m_\sigma^2(\sigma - v)^2 + m_{\eta'}^2{\eta'}^2 + O((\sigma - v)^3, {\eta'}^4)$
  - both masses of order Λ_{QCD}

In quark language

Classical symmetry

- $\psi \longrightarrow e^{i\phi\gamma_5/2}\psi$
- $\overline{\psi} \longrightarrow \overline{\psi}e^{i\phi\gamma_5/2}$

- mixes σ and η'
  - $\sigma \longrightarrow \sigma\cos(\phi) + \eta'\sin(\phi)$
  - $\eta' \longrightarrow -\sigma\sin(\phi) + \eta'\cos(\phi)$

This symmetry is "anomalous"

- any valid regulator must break chiral symmetry
  - remnant of the breaking survives in the continuum

Variable change alters fermion measure

- $d\psi \longrightarrow |e^{-i\phi\gamma_5/2}| d\psi = e^{-i\phi \mathrm{Tr}\gamma_5/2} d\psi$

But doesn't $\mathrm{Tr}\gamma_5 = 0$???

Fujikawa: Not in the regulated theory!!!

- i.e. $\lim_{\Lambda\to\infty} \mathrm{Tr}\left(\gamma_5 e^{D^2/\Lambda^2}\right) \neq 0$

Dirac action $\overline{\psi}(D+m)\psi$

- $D^\dagger = -D$
- $[D, \gamma_5]_+ = 0$

Use eigenstates of D to define $\mathrm{Tr}\gamma_5$

- $D|\psi_i\rangle = \lambda_i |\psi_i\rangle$
- $\mathrm{Tr}\gamma_5 = \sum_i \langle\psi_i|\gamma_5|\psi_i\rangle$

Index theorem

- with gauge winding ν, D has ν zero modes $D|\psi_i\rangle = 0$
  - modes are chiral: $\gamma_5|\psi_i\rangle = \pm|\psi_i\rangle$
  - $\nu = n_+ - n_-$

Non-zero eigenstates in chiral pairs

- $D|\psi\rangle = \lambda|\psi\rangle$
- $D\gamma_5|\psi\rangle = -\lambda\gamma_5|\psi\rangle = \lambda^*\gamma_5|\psi\rangle$

Space spanned by $|\psi\rangle$ and $|\gamma_5\psi\rangle$ gives no contribution to $\mathrm{Tr}\gamma_5$

- $\langle\psi|\gamma_5|\psi\rangle = 0$ when $\lambda \neq 0$
- only the zero modes count!

$$\boxed{\mathrm{Tr}\gamma_5 = \sum_i \langle\psi_i|\gamma_5|\psi_i\rangle = \nu}$$

Where did the opposite chirality go?

- continuum: lost at “infinity”
  - opposite chirality states “above the cutoff”
- overlap: modes on opposite side of unitarity circle
  - $D\gamma_5 = -\hat{\gamma}_5 D \qquad \mathrm{Tr}\ \hat{\gamma}_5 = 2\nu$
- Wilson: real eigenvalues in doubler region

This phenomenon involves both short and long distances

- zero modes compensated by modes lost at the cutoff
- cannot ignore instanton physics at short distances

Cannot uniquely separate perturbative and non-perturbative effects

- small instantons can “fall through the lattice”
- scheme and scale dependent

Under the transformation

- $\psi \longrightarrow e^{i\phi\gamma_5/2}\psi$
- $\overline{\psi} \longrightarrow \overline{\psi}e^{i\phi\gamma_5/2}$

Regulated fermion measure changes by $e^{-i\phi \mathrm{Tr}\gamma_5} = e^{-i\phi\nu}$

- changes weighting of gauge configurations with winding
  - non-zero ϕ introduces a sign problem for Monte Carlo

Note: here ϕ is the conventional Θ/N_f

- each flavor contributes equally
- $\mathrm{Tr}\gamma_5 = N_f\nu$

Expanding in smooth gauge fields

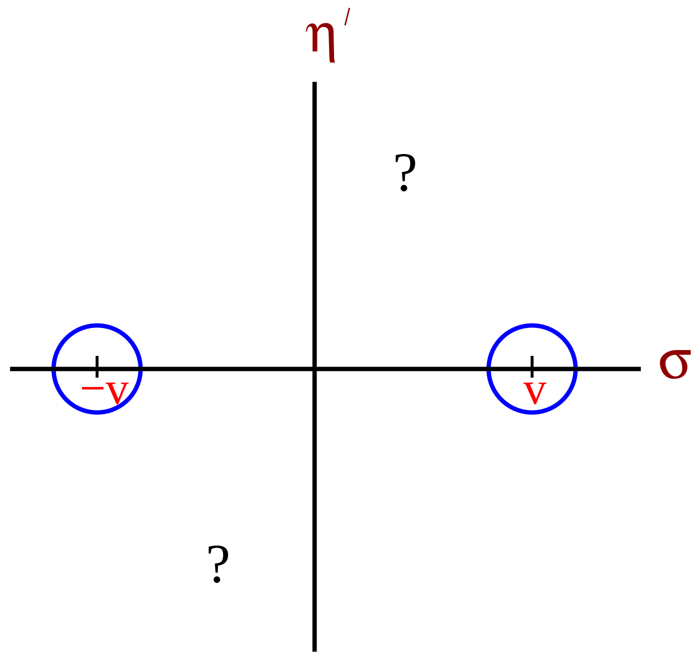
- $\nu = \mathrm{Tr}\ \gamma_5 e^{D^2/\Lambda^2} = \mathrm{Tr}\ e^{-\partial^2/\Lambda^2} \left(\frac{g^2}{\Lambda^4} F_{\mu\nu} \tilde{F}_{\mu\nu} + O(\Lambda^{-5}) \right)$
- trace over space, spinor, and color indices
- $e^{-\partial^2/\Lambda^2}$ "heat kernel regulator" for space
  - $\mathrm{Tr}_x (e^{-\partial^2/\Lambda^2} f(x)) = \frac{\Lambda^4}{16\pi^2} \int d^4x f(x)$

$$\nu = \frac{g^2}{16\pi^2} \int d^4x \mathrm{Tr}_c F_{\mu\nu} \tilde{F}_{\mu\nu}$$

- If $F = \tilde{F}$ then $S = \int d^4x \frac{1}{2} F^2 = \frac{8\pi^2}{g^2} \nu$
  - classical "instanton" action

Back to effective field language

At least two minima in the σ, η' plane



Question: do we know anything about the potential elsewhere in the σ, η' plane?

Yes!

- there are actually N_f equivalent minima

Define $\psi_L = \frac{1+\gamma_5}{2}\psi$

Singlet rotation $\psi_L \longrightarrow e^{i\phi}\psi_L$

- not a good symmetry for generic ϕ

Flavored rotation $\psi_L \longrightarrow g_L\psi_L = e^{i\phi_\alpha\lambda_\alpha}\psi_L$

- is a symmetry for $g_L \in SU(N_f)$

For special discrete values of ϕ these rotations can cross

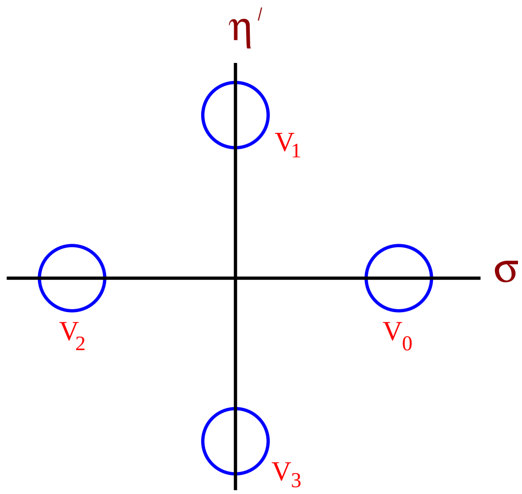
- $g = e^{2\pi i/N_f} \in Z_{N_f} \subset SU(N_f)$

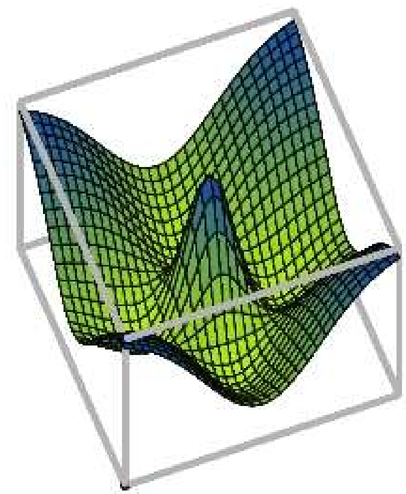
A valid discrete singlet symmetry:

$$\begin{aligned} \sigma &\longrightarrow \sigma\cos(2\pi/N_f) + \eta'\sin(2\pi/N_f) \\ \eta' &\longrightarrow -\sigma\sin(2\pi/N_f) + \eta'\cos(2\pi/N_f) \end{aligned}$$

$V(\sigma, \eta')$ has a Z_{N_f} symmetry

- N_f equivalent minima in the (σ, η') plane
- $N_f = 4$:



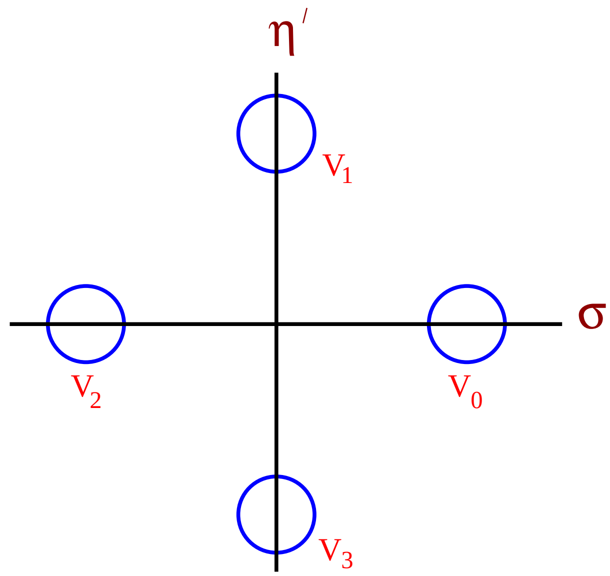


At the chiral lagrangian level

- Z_N is a subgroup of both $SU(N)$ and $U(1)$

At the quark level

- measure gets a contribution from each flavor ('t Hooft vertex)
- $\psi_L \to e^{2\pi i/N_f} \psi_L$ is a symmetry



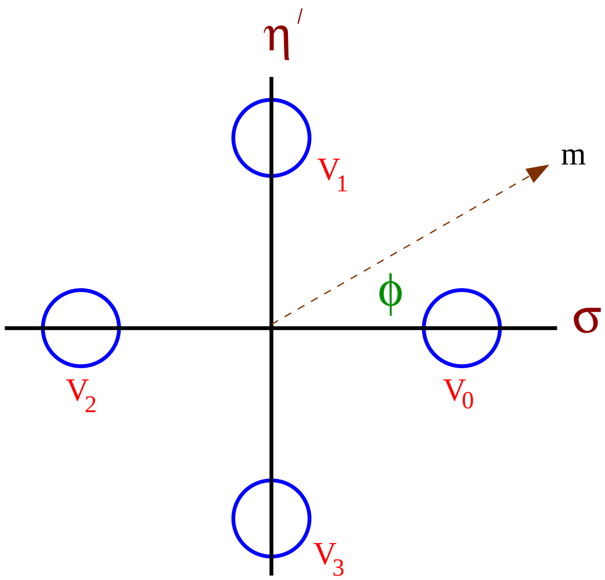
Mass term $m\overline{\psi}\psi$ tilts effective potential

- picks one vacuum as the lowest
- in n'th minimum $m_\pi^2 \sim m\cos(2\pi n/N_f)$
  - highest minima are unstable in the π_α direction
  - multiple meta-stable minima when $N_f > 4$

Anomalous rotation of the mass term

- $m\overline{\psi}\psi \to m\cos(\phi)\overline{\psi}\psi + im\sin(\phi)\overline{\psi}\gamma_5\psi$
- twists tilt away from the σ direction

An inequivalent theory!

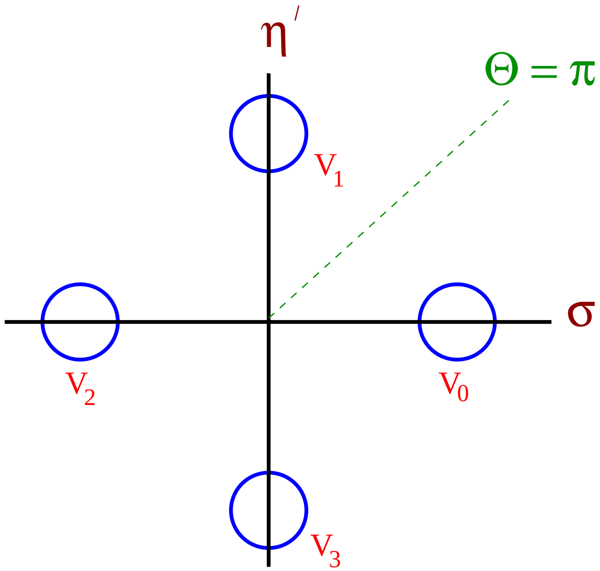


- as ϕ increases, vacuum jumps from one minimum to the next

Here each flavor has been given the same phase

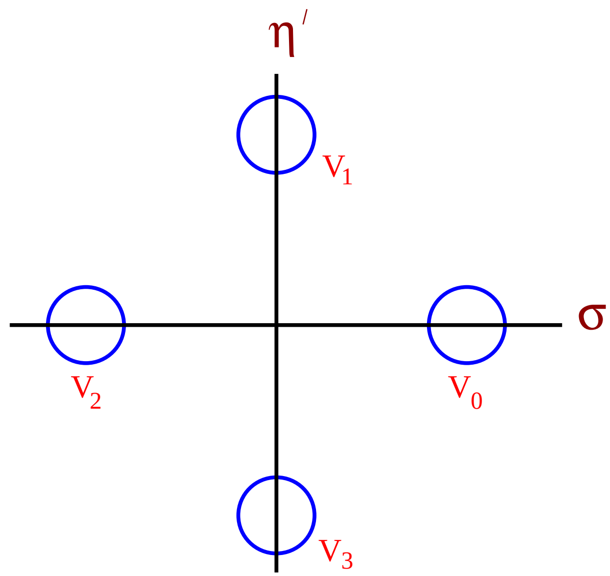
- Conventional notation uses $\Theta = N_f \phi$
- Z_{N_f} symmetry implies 2π periodicity in Θ

Degenerate light quarks $\Rightarrow$ first order transition at $\Theta = \pi$



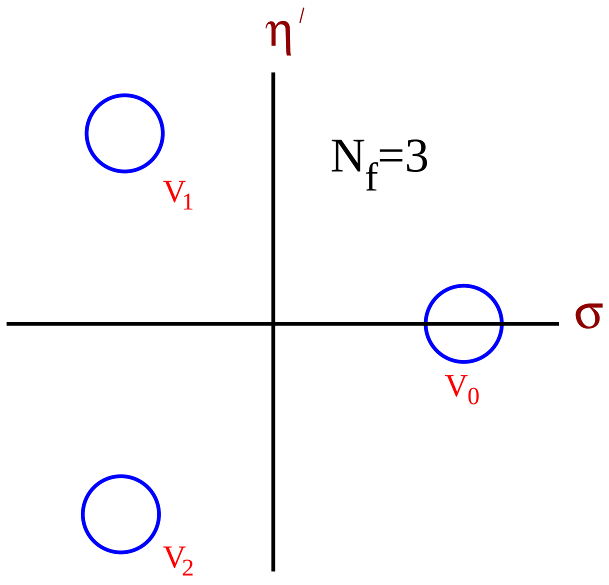
Discrete symmetry in mass parameter space $m \to m \exp\left(\frac{2\pi i \gamma_5}{N_f}\right)$

- for $N_f = 4$:
  - $m\overline{\psi}\psi$ and $im\overline{\psi}\gamma_5\psi$ mass terms give equivalent theories
  - true if and only if N_f is a multiple of 4



Odd number of flavors, $N_f = 2N + 1$

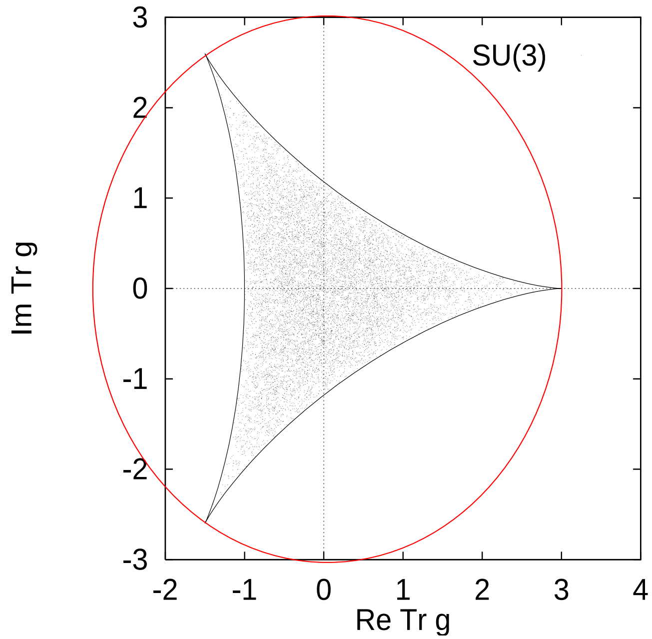
- -1 is not in $SU(2N+1)$
- $m > 0$ and $m < 0$ not equivalent!
- $m < 0$ represents $\Theta = \pi$
  - an inequivalent theory
  - spontaneous CP violation: $\langle \eta' \rangle \neq 0$

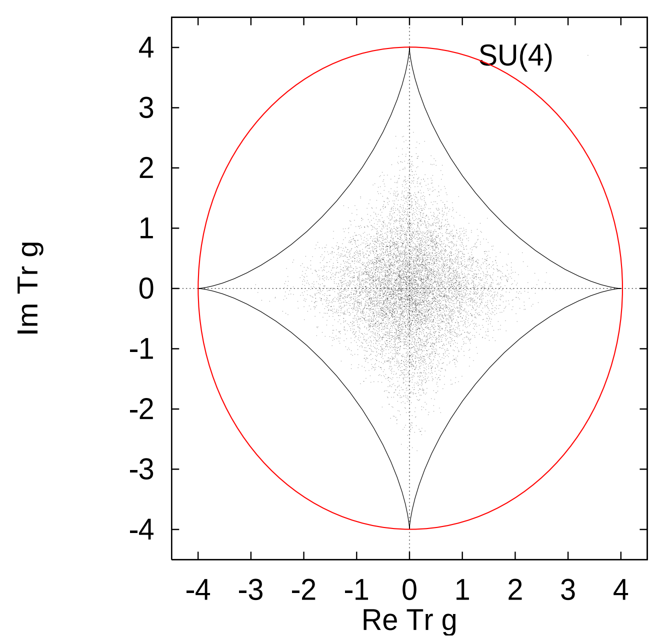


Inequivalent theories can have identical perturbative expansions!

Center of $SU(N_f)$ is a subgroup of $U(1)$

- 10,000 random $SU(3)$ and $SU(4)$ matrices:

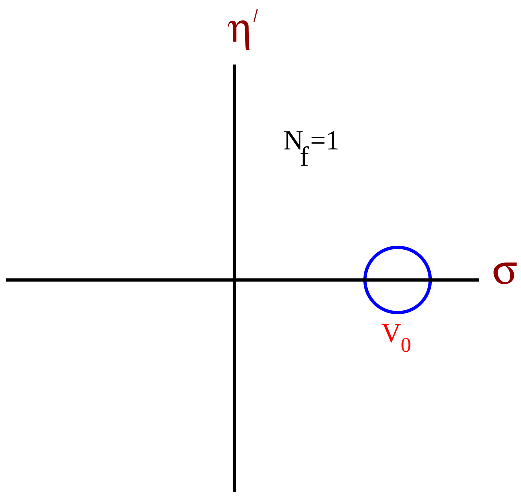




- region for $SU(3)$ bounded by $\exp(i\phi\lambda_8)$
- all $SU(N)$ points enclosed by the $U(1)$ circle $e^{i\phi}$
  - boundary reached at center elements

$N_f = 1$: No chiral symmetry at all!

- unique vacuum
- $\langle \overline{\psi}\psi \rangle \sim \langle \sigma \rangle \neq 0$ from 't Hooft vertex
- not a spontaneous symmetry breaking



No singularity at $m = 0$

- $m = 0$ not protected

For small mass

- no first order transition at $\Theta = \pi$
- larger masses?

Instanton effects scheme dependent (“renormalon” ambiguity)

- (Θ, m) singular coordinates when $N_f = 1$
  - $(\mathrm{Re}\, m, \mathrm{Im}\, m)$ better
- rough gauge configurations: ambiguous winding number
  - overlap operator not unique: depends on “domain wall height”

$m = 0$ for a non-degenerate quark is an ambiguous concept

$N_f = 0$: pure gauge theory

- $\Theta = \pi$ behavior unknown

When is rooting OK?

Starting with four flavors

- can we adjust N_f using

$$\begin{vmatrix} D+m & 0 & 0 & 0 \\ 0 & D+m & 0 & 0 \\ 0 & 0 & D+m & 0 \\ 0 & 0 & 0 & D+m \end{vmatrix}^{\frac{1}{4}} = |D+m| ?$$

A vacuous question outside the context of a regulator!

Rooting before removing regulator OK if

- regulator breaks anomalous symmetries on each factor
  - i.e. four copies of the overlap operator: $D\gamma_5 = -\hat{\gamma}_5 D$
  - winding ν from $\mathrm{Tr}\hat{\gamma}_5 = 2\nu$

Forcing Z_{N_f} symmetry with regulator in place

$$\begin{vmatrix} D + me^{\frac{i\pi\gamma_5}{4}} & 0 & 0 & 0 \\ 0 & D + me^{\frac{-i\pi\gamma_5}{4}} & 0 & 0 \\ 0 & 0 & D + me^{\frac{3i\pi\gamma_5}{4}} & 0 \\ 0 & 0 & 0 & D + me^{\frac{-3i\pi\gamma_5}{4}} \end{vmatrix}$$

- maintains $m \longrightarrow me^{i\pi\gamma_5/2}$ symmetry
  - permutes flavors

Four one-flavor theories with different values of Θ

- theta cancels out for the full four flavor theory

Rooting averages four inequivalent theories: NOT OK

- $(|D + M_1||D + M_2|)^{1/2} \neq \left|D + \sqrt{M_1 M_2}\right|$

Staggered fermions

- regulator maintains one exact chiral symmetry
  - $\left|D_s + m\right| = \left|D_s + me^{i\gamma_5\phi}\right|$
  - OK since actually a flavored symmetry
- separate into four “effective tastes” D_i
  - two tastes of each chirality

$$\left|D_s + me^{i\gamma_5\phi}\right| \sim \left|\begin{matrix} D_1+me^{i\phi\gamma_5} & 0 & 0 & 0 \\ 0 & D_2+me^{-i\phi\gamma_5} & 0 & 0 \\ 0 & 0 & D_3+me^{i\phi\gamma_5} & 0 \\ 0 & 0 & 0 & D_4+me^{-i\phi\gamma_5} \end{matrix}\right|$$

  - plus “taste mixing” (not the crucial issue)

Rooting to get to one flavor NOT OK

- rooting does not remove the Z_4
- tastes are not equivalent
  - rooting averages inequivalent theories

Rooted staggered fermions are not QCD!

Extra minima from Z_{N_f}

- expected to drive η' mass down
  - testable but difficult

Summary

QCD with N_f massless flavors has a discrete Z_{N_f} chiral symmetry

- flavor singlet

Associated with a first order transition at $\Theta = \pi$ when $m \neq 0$

Sign of mass significant for N_f odd

- not seen in perturbation theory

No symmetry for $N_f = 1$

- $m = 0$ unprotected

Structure inconsistent with rooted staggered quarks

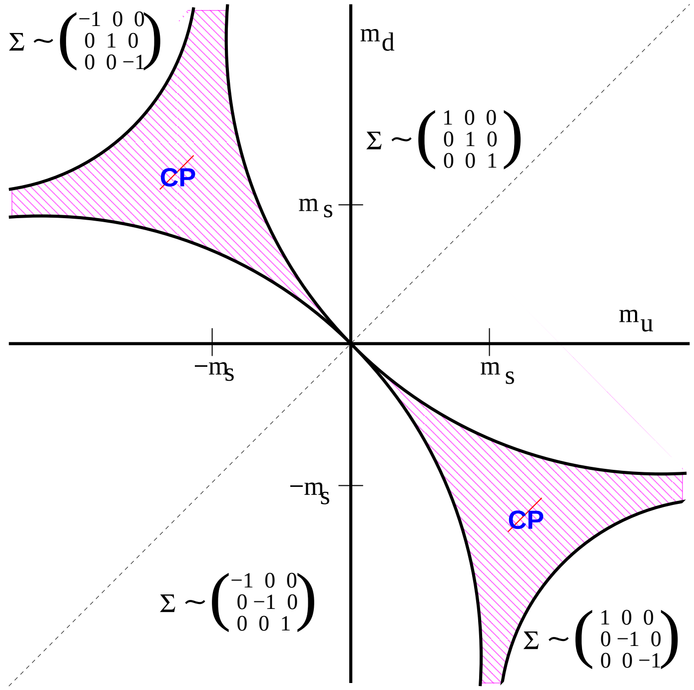
References:

- Ann. Phys. 324 (2009), 1573: arXiv:0901.0150
- arXiv:0909.5101

Extra slides: generalized quark masses

General masses can give intricate phase diagrams:

- three flavors
- fixed m_s
- vary m_u, m_d
- transition along $m_u = -\frac{m_d m_s}{m_d + m_s}$



Generalized mass term

- Renormalizable: look at fermion bilinears
- Lorentz invariant

Two flavors:

- $m_s\, \overline{\psi}\psi + m_v\, \overline{\psi}\tau_3\psi + im_3\, \overline{\psi}\gamma_5\psi + im_4\, \overline{\psi}\gamma_5\tau_3\psi$
  - m_s average quark mass, isoscalar
  - m_v quark mass difference, isovector
  - m_3 CP violating
  - m_4 “twisted mass” (useful with Wilson fermions)

These are not independent

- Fermion kinetic term symmetric under $\psi \longrightarrow e^{i\phi\gamma_5\tau_3}\psi$
- mixes m_s with m_4 and m_v with m_3

Can set any single $m_i = 0$ and another $m_j > 0$

- Choose convention $m_4 = 0$ and $m_s > 0$
  - $m_s\, \overline{\psi}\psi + m_v\, \overline{\psi}\tau_3\psi + i m_3\, \overline{\psi}\gamma_5\psi$

2 flavor QCD depends on three independent quark mass parameters

- $m_{\pi_\pm}^2 \sim m_s$
- $m_{\pi_\pm}^2 - m_{\pi_0}^2 \sim m_v^2$ (+EM effects)
- neutron electric dipole moment $\sim m_3$

Strong CP problem: why is m_3 experimentally extremely small?

- unification with CP violating weak interactions?

Symmetries make each term multiplicatively renormalizable

- CP symmetry protects m_3
- Isospin protects m_v
- Flavored chiral symmetry protects m_s

m_s and m_v transform differently under isospin

- m_v/m_s renormalization can depend on scheme
  - m_p, $m_{\pi_\pm}$ and m_{π_0} constant
  - m_p, $m_{\pi_\pm}$ and m_v/m_s constant
- not equivalent non-perturbatively (‘‘renormalon ambiguity’’)

Renormalization group interpretation

$$a\frac{dm}{da} = m\gamma(g) = m(\gamma_0 g^2 + \gamma_1 g^4 + \ldots) + \text{non-perturbative}$$

- non-perturbative part vanishes faster in g than any power

Bare quark masses run to zero logarithmically with the cutoff

- $m = m_r g^{\gamma_0/\beta_0}(1 + O(g^2)) \to_{a\to 0} 0$

Renormalized quark mass $m_r = \lim_{a\to 0}\ m g^{-\gamma_0/\beta_0}$

- “integration constant” of RG equation

Numerical value of m_r depends on scheme

- ’t Hooft vertex contributes a non-perturbative part $\sim m^{N_f-1}$
  - not proporional to m for $N_f = 1$
  - $m_{\eta'} \propto \frac{1}{a} e^{-1/2\beta_0 g^2} g^{-\beta_1/\beta_0^2}$
  - similar form can appear in RG equation

Example new scheme:

- $\tilde{a} = a$
- $\tilde{g} = g$
- $\tilde{m} = m - m_r g^{\gamma_0/\beta_0} \times \frac{e^{-1/2\beta_0 g^2} g^{-\beta_1/\beta_0^2}}{\Lambda a}$
- on RG trajectory the last factor approaches unity

Non-perturbative redefinition of parameters makes

- $\tilde{m}_r \equiv \lim_{a \to 0} \tilde{m} \tilde{g}^{-\gamma_0/\beta_0} = m_r - m_r = 0$
- $m_u = 0$ is not a solution to the strong CP problem

With degenerate quarks $m_\pi = 0$ defines $m = 0$