Self-Similarity and Power Asymptotics for Families of Stochastic Processes

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1. Ordinary (strict) self-similarity

- A stochastic process $X(\bullet)$ is H-SS with Hurst exponent $H \in (0,1)$ if

$$X(ct) \overset{d}{=} c^{H} X(t) \text{ for } t \geq 0,\ c > 0$$

Note: $\overset{d}{=}$ *denotes equality of the finite-dimensional distributions.*

- Main examples of strictly H-SS processes:
  - Fractional Brownian motion **(finite variance)**
  - Non-Gaussian α-stable processes **(infinite variance)**

- There is a lack of good H-SS processes with finite variance.

2. Strict self-similarity for processes with drift

- Stochastic process $X(\mu; \bullet)$ with drift $\mu \in \mathrm{R}$

$$X(\mu; t) = X(t) + \mu t$$

- The process $X(\bullet)$ is H-SS iff

$$X(ct) + \mu c^{H-1} ct \overset{d}{=} c^H X(t) + c^H \mu t \text{ for } t \geq 0$$

or

$$X(\mu c^{H-1}; ct) \overset{d}{=} c^H X(\mu; t) \text{ for } t \geq 0.$$

- Motivates an extended definition of self-similarity.

3. General self-similarity for families of processes

- The family X is called **general sense** H-SS with $H \in \mathrm{R}$ if

$$X(\mu c^{H-1}; ct) \overset{d}{=} c^H X(\mu; t) \text{ for } t \geq 0$$

for all $c > 0$ and $\mu \in \overline{\Omega} \subseteq \mathrm{R} \cup \{\infty\}$.

- Fixed point of **renormalization group**

$$c^{-H} X(\mu c^{H-1}; ct) \overset{d}{=} X(\mu; t) \text{ for all } c > 0.$$

- Assume **stationary increments** (SI)

$$X(\mu; s+t) - X(\mu; s) \overset{d}{=} X(\mu; t) - X(\mu; 0) \text{ for } t \geq 0.$$

- In general $\mu := \mathrm{E}X(\mu; 1)$ is a **rate parameter**,

$$\mathrm{E}X(\mu; t) = \mu t \text{ for all } t \geq 0 \text{ and } \mu \in \overline{\Omega}.$$

4. Covariance structure for self-similar families

- Define **variance function** for H-SSSI family X by

$$V(\mu) := \mathrm{Var} X(\mu; 1) \text{ for } \mu \in \Omega = \mathrm{int}\overline{\Omega}.$$

- Using $X(\mu; t) \overset{d}{=} t^H X(\mu t^{1-H}; 1)$ we obtain the **variance** in general

$$V_H(\mu; t) := \mathrm{Var} X(\mu; t) = t^{2H} V(\mu t^{1-H}).$$

- Using that $\mathrm{Var}\{X(\mu; t) - X(\mu; s)\} =$

$$\mathrm{Var} X(\mu; s) + \mathrm{Var} X(\mu; t) - 2\mathrm{Cov}\{X(\mu; s), X(\mu; t)\}$$

  the **covariance** becomes, for $s, t \geq 0$,

$$\mathrm{Cov}\{X(\mu; s), X(\mu; t)\} = \frac{1}{2}\{V_H(\mu; s) + V_H(\mu; t) - V_H(\mu; |t - s|)\}.$$

5. Power variance function

- **Power variance function** $V(\mu) = \mu^p$ for $\mu > 0$ gives

$$\mathrm{Var} X(\mu; t) = \mu^p t^{2-D}$$

  where $D \in [0, 2)$ is a **fractal dimension**

$$D = (H-1)(p-2) \overset{\text{FBM}}{=} 2(1-H).$$

- The **covariance** is, for $s, t \geq 0$,

$$\mathrm{Cov}\{X(\mu; s), X(\mu; t)\} = \frac{1}{2}\mu^p \{s^{2-D} + t^{2-D} - |t-s|^{2-D}\}.$$

- The increments $X(\mu; s+t) - X(\mu; s)$ are
  - **Negatively correlated** for $D > 1$.
  - **Uncorrelated** for $D = 1$.
  - **Positively correlated** for $D < 1$.

6. Families generated by exponential tilting

- Let $F_\mu^t :\sim X(\mu; t)$. Define **cumulant transform** κ by

$$\kappa(\theta) = \log \int_{\mathrm{R}} e^{\theta x} dF_1^1(x)$$

- Define F_μ^t by **exponential tilting**,

$$dF_\mu^1(x) = dF_1^1(x) e^{\theta x - \kappa(\theta)},$$

- Define **rate** $\mu = \dot{\kappa}(\theta)$ and **variance function**

$$V(\mu) = \ddot{\kappa}(\dot{\kappa}^{-1}(\mu)) \text{ for } \mu \in \Omega$$

- The **moment generating function** for general t is

$$\begin{aligned} \mathrm{E}\exp\{uX(\mu;t)\} &= \mathrm{E}\exp\{ut^H X(\mu t^{1-H}; 1)\} \\ &= \exp\{\kappa(ut^H + \theta') - \kappa(\theta')\}, \end{aligned}$$

where $\theta' = \dot{\kappa}^{-1}(\mu t^{1-H})$. Gives **exponential family** for each $t > 0$.

7. Hougaard Lévy processes

- **Hougaard Lévy process** $S_p(\mu;t)$ is an exponential family of Lévy processes with power variance function $V(\mu) = \mu^p$ for $\mu > 0$.
- Define **Tweedie distribution** with mean μ and variance $t^{-1}\mu^p$

$$\mathrm{Tw}_p(\mu,t) :\sim t^{-1}S_p(\mu;t)$$

- **Scale equivariance**

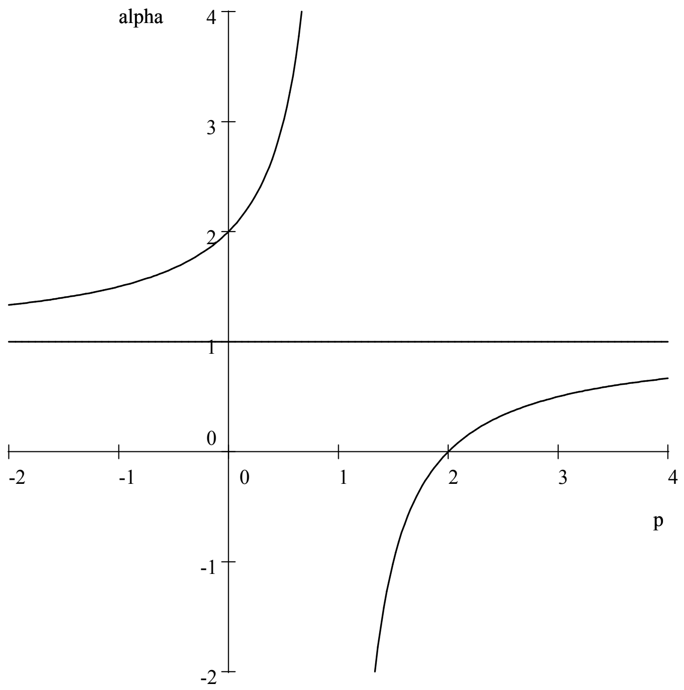
$$c\mathrm{Tw}_p(\mu,t) = \mathrm{Tw}_p(c\mu, c^{p-2}t)$$

implies

$$S_p(\mu;t) \sim t\mathrm{Tw}_p(\mu,t) = \mathrm{Tw}_p(t\mu, t^{p-1})$$

- Define the parameter α by

$$\alpha = 1 + (1-p)^{-1}$$



$$\alpha = 1 + (1 - p)^{-1}$$

- **Theorem:** The Hougaard family $S_p(\mu; t)$ is H-SSSI for $p \neq 2$ with

$$H = 1/\alpha, \text{ where } H = 0 \Leftrightarrow \alpha = -\infty$$

8. Examples of Hougaard Lévy processes, $H = 1/\alpha$

- $H = 0$ ($\alpha = -\infty$): **Poisson process** $S_1(\mu; t)$

  0-SSSI property:

  $$S_1(\mu; t) \overset{d}{=} S_1(1; \mu t).$$

- $H < 0$ ($\alpha < 0$): **Compound Poisson processes** $(1 < p < 2)$

  $$S_p(\mu; t) = \sum_{i=1}^{S_1(\mu_1;t)} X_i(\mu/\mu_1; -\alpha)$$

  - X_i are i.i.d. gamma variables with index $-\alpha > 0$
  - $S_1(\mu_1; t)$ is a Poisson process with rate

  $$\mu_1 = \frac{\alpha - 1}{\alpha} \mu^{\alpha/(\alpha-1)}$$

- Example: $H = \alpha = -1$: Exponential jumps.

Hougaard Lévy processes II

- $H = ½$ ($\alpha = 2$): **Brownian motion with drift** $\mu \in$ R

$$S_0(\mu; t) = B(t) + \mu t \qquad \text{for } t \geq 0$$

- ½ $< H < 1$ ($1 < \alpha < 2$): $S_p(\mu; t)$ is an **exponential tilting** of an **extreme** α**-stable Lévy process** $S_p(0; t)$ $\qquad (p < 0)$ spectrally negative, support R.
- $H > 1$ ($0 < \alpha < 1$): $S_p(\mu; t)$ is an **exponential tilting** of a **positive** α**-stable Lévy process** $S_p(\infty; t)$ $\qquad (p > 2)$
- $H = 2$ ($\alpha = 1/2$): The **Inverse Gaussian process** $S_3(\mu; t)$ is an **exponential tilting** of a **positive ½-stable Lévy process** $S_3(\infty; t)$.

Hougaard Lévy processes III

- $H = \infty$ ($\alpha = 0$): The **gamma process** $S_2(\mu; t)$ satisfies a scaling property

  $$cS_2(\mu; t) \overset{d}{=} S_2(c\mu; t)$$

  "kind of ∞-SSSI".

- $H = \alpha = 1$: $S_\infty(\mu; t)$ an **exponential tilting** of a 1-**stable Lévy process** $S_\infty(0; t)$.

  Satisfies an extended kind of 1-SSSI property

  $$S_\infty(\mu + \log c; ct) \overset{d}{=} c\{S_\infty(\mu; t) + t \log c\}.$$

  Gives an **exponential variance function** ($p = \infty$)

  $$V(\mu) = e^\mu \text{ for } \mu \in \mathrm{R}.$$

9. Fractional Hougaard motions (work in progress)

- Let $Z_\alpha(t)$ be an **extreme α-stable Lévy process**.
- Define the **linear fractional extreme α-stable motion** $Z_{\alpha,H}(t)$ by

$$Z_{\alpha,H}(t) = \int_{-\infty}^{\infty} w_h(t,u)\, dZ_\alpha(u) \text{ for } t \geq 0,$$

where $h = H - 1/\alpha,\ 0 < H < 1$ and

$$w_h(t,u) = \begin{cases} (t-u)^h - (-u)^h & \text{for} \quad u < 0 \\ (t-u)^h & \text{for} \quad 0 \leq u < t \\ 0 & \text{for} \quad t \leq u. \end{cases}$$

- **Fractional Hougaard motion** is an exponential tilting of $Z_{\alpha,H}(t)$ H-SSSI with power variance function $V(\mu) = \mu^p$.
- Special case $p = 0$ ($\alpha = 0$) is the **fractional Brownian motion**.

10. Central Limit Theorem—strictly stable case

- α**-stable convergence** for $0 < \alpha < 1$ or $1 < \alpha \leq 2$
  1. Stability of strictly α-stable Lévy process $Z_\alpha(t)$
  $$c^{-1/\alpha} Z_\alpha(ct) \overset{d}{=} Z_\alpha(t) \text{ for all } c > 0.$$
  2. Convergence to α-stable Lévy process
  $$c^{-1/\alpha} X(ct) \overset{d}{\to} Z_\alpha(t) \text{ as } c \to \infty.$$
- **Accommodate drift** in the notation,
$$X(\mu; t) = X(t) + \mu t \text{ and } Z_\alpha(\mu; t) = Z_\alpha(t) + \mu t.$$
- Rewrite α-stable convergence:
  1. Stability $\quad c^{-1/\alpha} Z_\alpha(\mu c^{1/\alpha - 1}; ct) \overset{d}{=} Z_\alpha(\mu; t).$
  2. Convergence
  $$c^{-1/\alpha} X(\mu c^{1/\alpha - 1}; ct) \overset{d}{\to} Z_\alpha(\mu; t) \text{ as } c \to \infty.$$

11. Hougaard convergence

- **Hougaard Lévy process** $S_p(\mu;t) \overset{d}{\sim} t\mathrm{Tw}_p(\mu,t)$ with $\alpha = 1 + (1-p)^{-1}$

  1. Stability
  $$c^{-1/\alpha} S_p(\mu c^{1/\alpha-1}; ct) \overset{d}{=} S_p(\mu;t).$$
  2. Convergence of exponential family X
  $$c^{-1/\alpha} X(\mu c^{1/\alpha-1}; ct) \overset{d}{\to} S_p(\mu;t) \text{ as } c \to \infty.$$
  3. Domain of attraction defined by
  $$V(\mu) \sim \mu^p \text{ as } \mu \downarrow 0 \text{ or } \mu \to \infty,$$
  depending on whether $c^{1/\alpha-1}$ goes to 0 or ∞.

12. Examples of Hougaard convergence

- $H = 0$ $(\alpha = -\infty)$: **Poisson convergence** to $S_1(\mu; t)$
  1. Stability of Poisson process

$$S_1(\mu c^{-1}; ct) \overset{d}{=} S_1(\mu; t).$$

  2. Convergence to Poisson process

$$X(\mu c^{-1}; ct) \overset{d}{\to} S_1(\mu; t) \text{ as } c \to \infty.$$

  3. Domain of attraction:

$$V(\mu) \sim \mu \text{ as } \mu \downarrow 0.$$

- Example: Bernoulli sequence X_i ~Bernoulli(μc^{-1})

$$X_1 + \cdots + X_{cn} \overset{d}{\to} S_1(\mu, n) \text{ as } c \to \infty.$$

Hougaard convergence

- $H < 0$ ($\alpha < 0$): **Compound Poisson convergence** $(1 < p < 2)$

  1. Stability

$$c^{-1/\alpha} S_p(\mu c^{1/\alpha - 1}; ct) \overset{d}{=} S_p(\mu; t).$$

  2. Convergence

$$c^{-1/\alpha} X(\mu c^{1/\alpha - 1}; ct) \overset{d}{\to} S_p(\mu; t) \text{ as } c \to \infty.$$

  3. Domain of attraction

$$V(\mu) \sim \mu^p \text{ as } \mu \downarrow 0.$$

- E.g. exponential compound Poisson convergence ($p = 1.5$). Comes up in Branching diffusion processes.

Hougaard convergence II

- $H = ½$ $(\alpha = 2)$: **Brownian motion with drift** $\mu \in \mathrm{R}$

  1. Stability

  $$c^{-1/2} S_0(\mu c^{-1/2}; ct) \overset{d}{=} S_0(\mu; t).$$

  2. Convergence

  $$c^{-1/2} X(\mu c^{-1/2}; ct) \overset{d}{\to} S_0(\mu; t) \text{ as } c \to \infty.$$

  3. Domain of attraction

  $$V(\mu) \sim \mu^0 = 1 \text{ as } \mu \downarrow 0.$$

- Classical central limit theorem.

Hougaard convergence II

- $\frac{1}{2} < H < 1$ $(1 < \alpha < 2)$: **Extreme α-stable** $S_p(\mu;t)$ $(p < 0)$ or $H > 1$ $(0 < \alpha < 1)$: **Positive α-stable** $S_p(\mu;t)$ $(p > 2)$
  1. **1.** Stability

$$c^{-1/\alpha} S_p(\mu c^{1/\alpha - 1}; ct) \overset{d}{=} S_p(\mu;t).$$

  2. **2.** Convergence

$$c^{-1/\alpha} X(\mu c^{1/\alpha - 1}; ct) \overset{d}{\to} S_p(\mu;t) \text{ as } c \to \infty.$$

  3. **3.** Domain of attraction

$$V(\mu) \sim \mu^p \text{ as } \mu \downarrow 0 \text{ or } \mu \to \infty.$$

- E.g. Inverse Gaussian convergence $(p = 3,\ \mu \to \infty)$.

Hougaard convergence III

- $H = \infty$ $(\alpha = 0)$: **Gamma convergence** $S_2(\mu; t)$.
  1. **1.** Stability

$$c^{-1}S_2(\mu c; t) \overset{d}{=} S_2(\mu; t).$$

  2. **2.** Convergence

$$c^{-1}X(\mu c; t) \overset{d}{\to} S_2(\mu; t) \text{ as } c \downarrow 0 \text{ or } c \to \infty.$$

  3. **3.** Domain of attraction

$$V(\mu) \sim \mu^2 \text{ as } \mu \downarrow 0 \text{ or } \mu \to \infty.$$

Hougaard convergence III

- $H = \alpha = 1$: **Extreme 1-stable** $S_\infty(\mu; t)$
  1. Stability

     $$c^{-1} S_\infty(\mu + \log c; ct) - t \log c \overset{d}{=} S_\infty(\mu; t).$$

  2. Convergence

     $$c^{-1} X(\mu + \log c; ct) - t \log c \xrightarrow{d} S_\infty(\mu; t) \text{ as } c \to \infty.$$

  3. Domain of attraction

     $$V(\mu) \sim e^{\mu} \text{ as } \mu \to \infty.$$

13. Lamperti-type convergence

- Let $X(t)$ be a **stochastic process**. Let $A(c) \to \infty$ as $c \to \infty$.
  - Convergence

    $$A(c)^{-1}X(ct) \overset{d}{\to} S(t) \text{ as } c \to \infty$$

    implies that
  - $S(t)$ is **self-similar** and $A(c) = c^H L(c)$ is regularly varying.
- Process with drift: $X(\mu; t) = X(t) + \mu t$.
  - Convergence

    $$A(c)^{-1}X(\mu A(c)c^{-1}; ct) = A(c)^{-1}X(ct) + \mu t \overset{d}{\to} S(t) + \mu t$$

    implies that
  - $S(\mu; t) = S(t) + \mu t$ **is self-similar**.

14. Generalizing Lamperti convergence

- Let $X(\mu; t)$ be a **family of stochastic processes**. Let $A(c) \to \infty$ as $c \to \infty$.
  - Assume convergence

$$A(c)^{-1} X(\mu A(c) c^{-1}; ct) \overset{d}{\to} S(\mu; t)$$

  - **Does this imply that $S(\mu; t)$ is self-similar** and $A(c) = c^H L(c)$?

- True for $L(c)$ constant.

15. Main points

- Strict self-similarity and general self-similarity.
- Covariance structure determined by variance function $V(\mu)$.
- Power variance function $V(\mu) = \mu^p$.
- Exponential tilting.
- Hougaard Lévy processes and fractional Hougaard motion.
- Central limit theorem and Hougaard convergence.
- Domain of attraction: $V(\mu) \sim \mu^p$ as $\mu \downarrow 0$ or $\mu \to \infty$.
- Lamperti-type convergence.

16. References

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