Multi-moment maps

$\mathbb{C}P^3$ Journal Club

Thomas Bruun Madsen

20th November 2009

Geometry with torsion
- Strong KT manifolds
- Strong HKT geometry

Strong KT manifolds: a new classification result

Multi-moment maps
- Motivation
- Existence
- Examples

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Geometry with torsion

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$$c(X, Y, Z) = g\left(T(X, Y), Z\right).$$

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Put $\nabla = \nabla^{LC} + \frac{1}{2}T$ then

- ∇ is metric and has $(2,1)$-torsion $T^{\nabla} = T$;
- ∇ is unique metric connection with torsion $c = T^{\flat}$;
- ∇, ∇^{LC} have same geodesics ($\nabla_X X = \nabla^{LC}_X X$).

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Torsion geometry study of triplet (M, g, c) equipped with ∇. Is called **strong** if $dc = 0$.

What if we add a compatible complex structure?

Hermitian structure

Hermitian manifold consists of: (M, g) Riemannian and endomorphism $J\colon\ TM \to TM$ satisfying:

- $J^2 = -1$;
- $0 = [JX, JY] - J([JX, Y] + [X, JY]) - [X, Y]$;
- $g(JX, JY) = g(X, Y)$.

(M, g, J) is **Kähler** if the fundamental two-form $\omega = g(J\cdot, \cdot)$ is closed, i.e., $d\omega = 0$.

Kähler with torsion geometry

Theorem (Gauduchon, 1997)

Let (M, g, J) be a Hermitian manifold. Then there exists a unique Hermitian connection ∇^B which has skew-symmetric torsion. This connection is characterised by the formula

$$\nabla^B = \nabla^{LC} + \tfrac{1}{2} T^B,$$

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Study of Hermitian manifold with Bismut connection is **KT geometry**.

Definition

A Hermitian manifold (M, g, J) is called a **strong KT manifold**, briefly an **SKT manifold**, if

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How restrictive is this condition?...

Examples

Example

Kähler manifolds are precisely SKT manifolds with $c^B = 0$.

Example

(M^4, g, J) **compact Hermitian** 4-**manifold**. Exists unique (up to homothety) SKT metric in conformal class of g.

Examples

Example

G an even-dimensional compact Lie group:

- $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \bigoplus_{\alpha \in P} \mathfrak{g}_\alpha^{\mathbb{C}}$;
- $-B$ on $[\mathfrak{g}, \mathfrak{g}]$ extends to J-compatible inner product g on $\mathfrak{g}$;
- $\nabla_X Y = 0 \quad (X, Y \in \mathfrak{g})$.

∇ metric connection which preserves J and has torsion the closed three-form $-g([X, Y], Z)$.

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Proposition

Any even-dimensional compact Lie group *G admits a left-invariant SKT structure. Moreover each left-invariant complex structure on G admits a compatible invariant SKT metric.*

Thus SKT condition

- much less restrictive than being Kähler;
- however much more restrictive than KT condition.

Theorem (Fino-Parton-Salamon, 2004)

Let G be a simply-connected six-dimensional nilpotent real Lie group. Then G admits a left-invariant SKT structure if only if its Lie algebra $\mathfrak{g}$ is isomorphic to one of $(0,0,0,0,0,0)$, $(0,0,0,0,13+42,14+23)$, $(0,0,0,0,12,14+23)$, $(0,0,0,0,12,34)$, $(0,0,0,0,0,12)$.

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- For a 6-dimensional nilpotent Lie group with an invariant complex structure J, the SKT condition is satisfied either by all invariant Hermitian metrics g or by none. There are 13 algebras that admit invariant KT structure but no invariant SKT structure.

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- For a 6-dimensional nilpotent Lie group with an invariant complex structure J, the SKT condition is satisfied either by all invariant Hermitian metrics g or by none. There are 13 algebras that admit invariant KT structure but no invariant SKT structure.
- By Malcev's Theorem above Lie groups G admit co-compact discrete subgroups Γ, so $\Gamma \backslash G$ is nilmanifold.
- Similar calculations in dimension four give two algebras $(0,0,0,0)$, $(0,0,0,12)$ (precisely those with invariant complex structure).

Greater flexibility in SKT condition compared with Kähler implies lack of central results, e.g., have **no Calabi-Yau Theorem**. Best to date is:

Theorem (Tosatti & Weinkowe, 2009)

Let (M^{2n}, g, J) be a compact connected SKT manifold for which ω^2 is $\partial\bar{\partial}$-closed. For every smooth real function f on M there exists a unique smooth real function ϕ on M solving

$$(\omega + i\partial\bar{\partial}\phi)^n = \left(\frac{\int_M \omega^n}{\int_M e^f \omega^n}\right) e^f \omega^n, \ \textit{with}$$

$$\omega + i\partial\bar{\partial}\phi > 0, \sup_M \phi = 0.$$

In particular $\omega + i\partial\bar{\partial}\phi$ is an SKT form on M with the same volume as ω.

SKT geometry and bi-Hermitian geometry

SKT structures frequently appear in terms of **bi-Hermitian structures**: M is a manifold endowed with SKT structures (g, J_+) and (g, J_-) such that

$$(\bar{\partial}_+ - \partial_+)\omega_+ + (\bar{\partial}_- - \partial_-)\omega_- = 0.$$

By [Apostolov & Gualtieri] the above is equivalent to having a generalized Kähler structure on M.

Why strong KT geometry?

Quest for canonical choices of metric compatible with given complex structure.

- SKT **less restrictive** than being Kähler;
- SKT condition seems to **behave well**, e.g., blow-up of strong KT manifold at point or along compact complex submanifold again SKT;
- Good understanding of SKT geometry helpful in study of strong HKT geometry.

Almost hyperHermitian structure

Almost hyperHermitian manifold: (M, g) Riemannian with three almost complex structures I, J, K satisfying:

- $K = IJ = -JI$;
- $g(X, Y) = g(IX, IY) = g(JX, JY) = g(KX, KY)$.

Define $\omega_I = g(I\cdot, \cdot)$, $\omega_J = g(J\cdot, \cdot)$, $\omega_K = g(K\cdot, \cdot)$.

HKT manifold

Definition

An almost hyperHermitian manifold (M, g, I, J, K) is a **hyperKähler manifold with torsion**, briefly an **HKT manifold**, if

$$Id\omega_I = Jd\omega_J = Kd\omega_K. \tag{1}$$

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$$Id\omega_I = Jd\omega_J = Kd\omega_K. \qquad (1)$$

HKT condition implies integrability of I, J, K. Bismut connection for (g, I), (g, J), and (g, K) coincide. Torsion three-form is

$$c = -Id\omega_I = -Jd\omega_J = -Kd\omega_K.$$

Have **strong HKT** if $dc = 0$.

Example

Consider $SU(3)$. Choose following basis of its Lie algebra $\mathfrak{su}(3)$:

$$A_1 = i(E_{11} - E_{22}), \quad A_2 = i(E_{22} - E_{33})$$
$$B_{\ell m} = E_{\ell m} - E_{m\ell}, C_{\ell m} = i(E_{\ell m} + E_{m\ell}) \quad (1 \leq \ell < m \leq 3).$$

Denote its dual basis by $a_1, \ldots, c_{23}$. Minus Killing form gives inner product

$$\langle \cdot, \cdot \rangle = -a_1 a_2 + 2\left(a_1^2 + a_2^2 + b_{12}^2 + \ldots + c_{23}^2\right).$$

Orthogonal (Joyce) decomposition

$$\mathfrak{su}(3) = \mathbb{R} \oplus \mathfrak{d} \oplus \mathfrak{f}.$$

$\mathbb{R} = \operatorname{span}\{A_1 + 2A_2\}$, $\mathfrak{d} = \operatorname{span}\{A_1, B_{12}, C_{12}\}$,
$\mathfrak{f} = \operatorname{span}\{B_{13}, B_{23}, C_{13}, C_{23}\}$.

Define action of I, J, K by taking highest root $\mathfrak{su}(2)$ ($\mathfrak{d}$) letting it act on complement ($\mathbb{R} \oplus \mathfrak{f}$), e.g.

$$I\left(\tfrac{1}{\sqrt{3}}(A_1 + 2A_2)\right) = A_1,\; I \;\text{ acts as }\; \tfrac{1}{2}\,\mathsf{ad}_{A_1} \;\text{ on }\; A_1^{\perp} \cap \mathfrak{d}$$

$$\mathsf{ad}_{A_1} \;\text{ on }\; \mathfrak{f}.$$

Left translating above gives almost hyperHermitian structure on $SU(3)$. Can verify that

$$\begin{aligned} -c = Id\omega_I = Jd\omega_J = Kd\omega_K &= a_1(2b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23}) \\ &- a_2(b_{12}c_{12} - b_{13}c_{13} - 2b_{23}c_{23}) - b_{23}c_{12}c_{13} - b_{13}c_{12}c_{23} \\ &- b_{12}c_{13}c_{23} - b_{12}b_{13}b_{23}. \end{aligned}$$

Gives **strong HKT** structure on $SU(3)$.

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Motivation

Classification of **invariant SKT structures** on six-dimensional nilpotent Lie groups. **Solvable Lie groups**?...Start in dimension four.

Invariant SKT structure

Identifications:

- $\mathfrak{g}$ left-invariant vector fields on G;
- $\mathfrak{g}^*$ left-invariant one-forms;
- $da(X, Y) = -a([X, Y])$, $a \in \mathfrak{g}^*$, $X, Y \in \mathfrak{g}$).

Left-invariant almost Hermitian structure on G determined by inner product g on $\mathfrak{g}$ and g-orthogonal linear endomorphism $J\colon \mathfrak{g} \to \mathfrak{g}$ with $J^2 = -1$. **SKT condition** requirement that

- J be integrable, $d\Lambda^{1,0} \subseteq \Lambda^{2,0} + \Lambda^{1,1}$;
- $d\omega(J\cdot, J\cdot, J\cdot)$ is in the kernel of d.

Solvable Lie algebras

$\Lambda^* \mathfrak{g}^*$ exterior algebra on $\mathfrak{g}^*$ and $\mathcal{I}(A)$ ideal in $\Lambda^* \mathfrak{g}^*$ generated by subset A. $\mathfrak{g}$ **solvable** means (dually):

Lemma

A finite-dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if there are maximal subspaces $\{0\} = W_0 < W_1 < \cdots < W_r = \mathfrak{g}^*$ such that

$$dW_i \subseteq \mathcal{I}(W_{i-1})$$

for each i.

$\mathfrak{g}$ solvable Lie algebra of dimension four. (g, J) an integrable Hermitian structure on $\mathfrak{g}$ then there is orthonormal set $\{a, b\}$ in $\mathfrak{g}^*$ such that $\{a, Ja, b, Jb\}$ is basis for $\mathfrak{g}^*$ and either

- $\mathfrak{g}$ has **structural equations**

$$
\begin{aligned}
da = 0, d(Ja) = x_1 aJa, \\
db = y_1 aJa + y_2 ab + y_3 aJb + z_1 bJa + z_2 JaJb, \\
d(Jb) = u_1 aJa + u_2 ab + u_3 aJb + v_1 bJa + v_2 JaJb + w_1 bJb,
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or

$$da = 0,\ d(Ja) = x_1 aJa + x_2(ab + JaJb) + x_3(aJb + bJa) + y_2 bJb,$$
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- In first case, $\{a, Ja, b, Jb\}$ may be chosen orthonormal and $\omega = aJa + bJb$, omitting $\wedge$ signs.
- In latter case $\omega = aJa + bJb + t(ab + JaJb)$ for some $t \in (-1, 1)$.

Above necessary, not sufficient. Need also

- $d^2 = 0$;
- $d(b - iJb)^{2,0} = 0$;
- SKT condition.

Impose unpleasant constraints on coefficients in structural equations...

...helps to observe that Lie's Theorem implies:

Lemma

A finite-dimensional Lie algebra $\mathfrak{g}$ *is solvable if and only if its derived algebra* $\mathfrak{g}'$ *is nilpotent.*

So in dimension 4 we know $\mathfrak{g}'$ **is either Abelian or** $\mathfrak{h}_3$.

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Lemma

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So in dimension 4 we know $\mathfrak{g}'$ **is either Abelian or** $\mathfrak{h}_3$. Now structural equations can be analysed, and we obtain:

Theorem (TBM & Swann, 2009)

Let G be a simply-connected four-dimensional solvable real Lie group. Then G admits a left-invariant SKT structure if only if its Lie algebra $\mathfrak{g}$ is listed in the following table.

$\mathfrak{g}'$	$\mathfrak{g}$	dim	π_0	Kähler	$(b_1 \dots b_4)$
$\{0\}$	$\mathbb{R}^4$	0	1	✓	$(4,6,4,1)$
$\mathbb{R}$	$\mathbb{R} \times \mathfrak{h}_3$	0	1	×	$(3,4,3,1)$
	$\mathbb{R} \times \mathfrak{r}_{3,0}$	1	1	✓	$(3,3,1,0)$
$\mathbb{R}^2$	$\mathbb{R} \times \mathfrak{r}'_{3,0}$	1	1	✓	$(2,2,2,1)$
	$\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$	2	1	✓	$(2,1,0,0)$
$\mathbb{R}^3$	$\mathfrak{r}'_{4,\lambda,0}\,(\lambda > 0)$	1	2	✓	$(1,1,1,0)$
	$\mathfrak{r}_{4,-1/2,-1/2}$	1	1	×	$(1,0,1,1)$
	$\mathfrak{r}'_{4,2\lambda,-\lambda}\,(\lambda > 0)$	1	2	×	$(1,0,1,1)$
$\mathfrak{h}_3$	$\mathfrak{d}_4$	2	1	×	$(1,0,1,1)$
	$\mathfrak{d}_{4,2}$	2	1	✓	$(1,1,1,0)$
	$\mathfrak{d}'_{4,0}$	2	1	×	$(1,0,1,1)$
	$\mathfrak{d}_{4,1/2}$	1	1	✓	$(1,0,0,0)$
	$\mathfrak{d}'_{4,\lambda}\,(\lambda > 0)$	1	1	✓	$(1,0,0,0)$

Remarks/Consequences

Corollary

There are four-dimensional solvable complex Lie groups whose family of compatible invariant Hermitian metrics contains both SKT and non-SKT structures.

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There are four-dimensional solvable complex Lie groups whose family of compatible invariant Hermitian metrics contains both SKT and non-SKT structures.

Corollary

The four-dimensional solvable Lie algebras $\mathfrak{g}$ *that admit invariant complex structures but no compatible invariant SKT metric are:* $\mathbb{R} \times \mathfrak{r}_{3,1}$, $\mathbb{R} \times \mathfrak{r}'_{3,\lambda>0}$, $\mathfrak{aff}_{\mathbb{C}}$, $\mathfrak{r}_{4,1}$, $\mathfrak{r}_{4,\mu,\lambda}\,(\mu = \lambda \neq -\frac{1}{2}$ *or* $\mu \leqslant 1)$, $\mathfrak{r}'_{4,\mu,\lambda}$ $(\lambda \neq 0, -\mu/2)$, $\mathfrak{d}_{4,\lambda}$ $(\lambda \neq \frac{1}{2}, 2)$, $\mathfrak{h}_4$.

Remarks/Consequences

Corollary

Each unimodular solvable four-dimensional Lie group G with invariant SKT structure admits a compact quotient by a lattice.

Corollary

The only four-dimensional solvable Lie algebra that is strong HKT is $\mathbb{R}^4$, which is hyperKähler. The algebra $\mathfrak{d}_{4,1/2}$ admits both HKT and SKT structures; these structures are distinct. The remaining HKT algebras $\mathfrak{aff}_{\mathbb{C}}$ and $\mathfrak{r}_{4,1,1}$ do not admit invariant SKT structures.

Strategy for classifying SKT

Similar procedure expected to work in dimension six. But how do we attack classification of strong KT manifolds in general?

Are there any **tools** we can use?

Corollary

Each invariant SKT structure on a four-dimensional solvable Lie group G is invariant under a two-dimensional Abelian subgroup $H \leqslant G$.

Such a fibration can be investigated using so-called **multi-moment maps** (work in progress).

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Usual moment map

(M, ω) a **symplectic manifold**.

- Lie group G acts on M preserving symplectic form. Under appropriate circumstances, e.g., $H^1(\mathfrak{g}) = 0 = H^2(\mathfrak{g})$, exists a (unique) equivariant map

$$\mu \colon M \to \mathfrak{g}^* \quad \text{s.t.} \quad d\langle \mu, X \rangle = \xi_X \lrcorner\, \omega, \quad (X \in \mathfrak{g}).$$

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  Such a **moment map** is useful, e.g., in Kähler geometry, since fundamental two-forms are symplectic forms. For SKT manifolds $c^B \neq 0$ in general, so not symplectic. But $dc^B = 0$.

Can ω in moment map construction be replaced by closed invariant three-form c on M?...

Covariant moment maps

Definition (Cariñena et al, 1991)

Let M be a manifold and α a closed p-form on M. Let G be a Lie group acting on M preserving α. A **covariant moment map** is a section $\sigma \in \Gamma(\mathfrak{g}^* \otimes \Lambda^{p-2} T^* M)$ satisfying

$$d\langle \sigma, X \rangle = \xi_X \lrcorner \alpha \quad (X \in \mathfrak{g}).$$

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- Consistent with usual notion of moment map when $p = 2$. For $p = 4$ fits with notion of quaternion-Kähler moment map;
- Case $p = 3$ of interest. Do not want section as above, but (equivariant) map $M \to V_{\mathfrak{g}}$ into vector space depending only on $\mathfrak{g}$.

Calculation

$X, Y \in \mathfrak{g}$ s.t. $[X, Y] = 0$ then

$$0 = [\xi_Y, \xi_X] \lrcorner c = d(\xi_Y \lrcorner \xi_X \lrcorner c),$$

so if $b_1(M) = \dim H^1(M, \mathbb{R}) = 0$, we have a scalar function $\nu_{X,Y} \colon M \to \mathbb{R}$ satisfying

$$d\nu_{X,Y} = \xi_Y \lrcorner \xi_X \lrcorner c.$$

More generally consider $p = \sum_j X_j \wedge Y_j \in \mathcal{P}_{\mathfrak{g}} = \ker\big(L\colon \Lambda^2\,\mathfrak{g} \to \mathfrak{g}\big)$, thus $0 = L(p) = \sum_j [X_j, Y_j]$. We get as before $\nu_p\colon M \to \mathbb{R}$ such that

$$d\nu_p = \sum_j \xi_{Y_j} \lrcorner\, \xi_{X_j} \lrcorner\, c = \xi_p \lrcorner\, c.$$

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$$d\nu_p = \sum_j \xi_{Y_j} \lrcorner \xi_{X_j} \lrcorner c = \xi_p \lrcorner c.$$

Module $\mathcal{P}_{\mathfrak{g}}$ so natural that one would expect much is known...

Remark on module $\mathcal{P}_{\mathfrak{g}}$

...seems not to be the case. For instance have not found explicitly the following result (which follows from result of Wolf on isotropy irreducible spaces):

Theorem

Let $\mathfrak{g}$ be a compact simple Lie algebra. Then $\Lambda^2\,\mathfrak{g}$ splits into the following irreducible modules

$$\Lambda^2\,\mathfrak{g} = \mathfrak{g} \oplus \mathcal{P}_{\mathfrak{g}}.$$

Multi-moment maps

Definition (TBM, 2008)

Let M be a manifold and $c \in \Omega^3(M)$, $dc = 0$. Let G be a Lie group acting on M preserving c. A **multi-moment map** is an equivariant map $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$ such that

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Multi-moment maps

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Have thus way obtained (appropriate) equivariant map from M into representation $\mathcal{P}_{\mathfrak{g}}$...if multi-moment maps exist...

General existence

Theorem (TBM, 2009)

Let M be a connected manifold and $c \in \Omega^3(M)$, $dc = 0$. Let G be a Lie group acting on M and preserving c. A multi-moment map $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}$ exists if one of the following conditions is satisfied:

- $b_1(M) = 0$ *and* G **is compact***;*
- M **is compact** *orientable manifold with* $b_1(M) = 0$*, and* G *preserves a volumeform on* M*;*
- $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$*.*

Another frequently encountered situation, where moments map exist: M is simply-connected and G **is Abelian**.

Example 1

N a manifold. $M = \Lambda^2 T^* N$ has canonical two-form β. At $\alpha \in M$ define

$$\beta_\alpha = \pi^*(\alpha), \quad \pi\colon M \to N \text{ is the bundle projection.}$$

Lie group action $G \times N \to N$ lifts to action $G \times M \to M$ preserving β and hence the closed three-form $c = d\beta$. A multi-moment map $\nu\colon M \to \mathcal{P}^*_{\mathfrak{g}}$ is given by

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Note similar construction exists in symplectic setting (and explains the name "moment" map).

Example 2

Consider the Lie group

$$G = \left\{ \begin{pmatrix} 1 & 0 & ye^{-x} & w \\ 0 & e^x & 0 & y \\ 0 & 0 & e^{-x} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : \ x, y, z, w \in \mathbb{R} \right\}$$

with corresponding Lie algebra $\mathfrak{d}_4$. For one concrete choice of invariant (non-Kähler) SKT structure (g, J) on G we find an Abelian group A acting on preserving the SKT structure such that there are maps $\nu_J, \nu\colon\ G \to \mathbb{R}$ given by

$$\nu_J = -\exp(-x), \quad \nu = z,$$

which are A-invariant and satisfy

$$d\nu_J = \xi_W \lrcorner \xi_Y \lrcorner d\omega, \quad d\nu = \xi_W \lrcorner \xi_Y \lrcorner Jd\omega.$$

These are multi-moment maps.

Applications

- Take $M = SU(3)$ with Joyce SHKT structure.
  $G = SU(3) \times U(1)$ acts on M preserving SHKT structure.
  Exists multi-moment map $\nu\colon M \to \mathcal{P}^*_{\mathfrak{g}}$ with image $\mathbb{C}P(2)$.
  Possible to **reconstruct the hypercomplex** structure on M from the information encoded in the multi-moment map.
  Leaves hope for treating (some) SHKT manifolds systematically using multi-moment maps.

Applications

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- (M^7, ϕ) is a G_2-manifold with an S^1-symmetry. Using Kähler reduction it has been shown that one sometimes get M as a T^2 bundle over a hyperKähler manifold. This **reduction** is more naturally described in terms of multi-moment maps.

Work to be done

Work in progress/to be done:

- use multi-moment maps as tools in **classification of (primarily) strong KT and strong HKT** manifolds;
- study **orbits** $\mathcal{O} \subset \mathcal{P}^*_{\mathfrak{g}}$;
- study **quotient constructions**;

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