

# Multi-moment maps

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## **Geometry with torsion**

Strong KT manifolds

Strong HKT geometry

## **Strong KT manifolds: a new classification result**

## **Multi-moment maps**

Motivation

Existence

Examples

## **References**

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## Geometry with torsion

$(M, g)$  a Riemannian manifold and  $c \in \Omega^3(M)$ . Define  $(2, 1)$ -tensor  $T$  by

$$c(X, Y, Z) = g(T(X, Y), Z).$$

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Put  $\nabla = \nabla^{LC} + \frac{1}{2}T$  then

- ▶  $\nabla$  is metric and has  $(2, 1)$ -torsion  $T^\nabla = T$ ;
- ▶  $\nabla$  is unique metric connection with torsion  $c = T^\flat$ ;
- ▶  $\nabla, \nabla^{LC}$  have same geodesics ( $\nabla_X X = \nabla_X^{LC} X$ ).

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Torsion geometry study of triplet  $(M, g, c)$  equipped with  $\nabla$ . Is called **strong** if  $dc = 0$ .

What if we add a compatible complex structure?

## Hermitian structure

**Hermitian manifold** consists of:  $(M, g)$  Riemannian and endomorphism  $J: TM \rightarrow TM$  satisfying:

- ▶  $J^2 = -1$ ;
- ▶  $0 = [JX, JY] - J([JX, Y] + [X, JY]) - [X, Y]$ ;
- ▶  $g(JX, JY) = g(X, Y)$ .

$(M, g, J)$  is **Kähler** if the fundamental two-form  $\omega = g(J\cdot, \cdot)$  is closed, i.e.,  $d\omega = 0$ .

## Kähler with torsion geometry

### Theorem (Gauduchon, 1997)

*Let  $(M, g, J)$  be a Hermitian manifold. Then there exists a unique Hermitian connection  $\nabla^B$  which has skew-symmetric torsion. This connection is characterised by the formula*

$$\nabla^B = \nabla^{LC} + \frac{1}{2} T^B,$$

*where  $c^B = (T^B)^\flat = d\omega(J\cdot, J\cdot, J\cdot)$ .*

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Study of Hermitian manifold with Bismut connection is **KT geometry**.

### Definition

A Hermitian manifold  $(M, g, J)$  is called a **strong KT manifold**, briefly an **SKT manifold**, if

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How restrictive is this condition?...

## Examples

### Example

**Kähler manifolds** are precisely SKT manifolds with  $c^B = 0$ .

### Example

$(M^4, g, J)$  **compact Hermitian 4-manifold**. Exists unique (up to homothety) SKT metric in conformal class of  $g$ .

## Examples

### Example

$G$  an even-dimensional compact Lie group:

- ▶  $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \bigoplus_{\alpha \in P} \mathfrak{g}_{\alpha}^{\mathbb{C}}$ ;
- ▶  $-B$  on  $[\mathfrak{g}, \mathfrak{g}]$  extends to  $J$ -compatible inner product  $g$  on  $\mathfrak{g}$ ;
- ▶  $\nabla_X Y = 0$  ( $X, Y \in \mathfrak{g}$ ).

$\nabla$  metric connection which preserves  $J$  and has torsion the closed three-form  $-g([X, Y], Z)$ .

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### Proposition

**Any even-dimensional compact Lie group  $G$  admits a left-invariant SKT structure. Moreover each left-invariant complex structure on  $G$  admits a compatible invariant SKT metric.**

Thus SKT condition

- ▶ much less restrictive than being Kähler;
- ▶ however much more restrictive than KT condition.

### Theorem (Fino-Parton-Salamon, 2004)

*Let  $G$  be a simply-connected six-dimensional nilpotent real Lie group. Then  $G$  admits a left-invariant SKT structure if and only if its Lie algebra  $\mathfrak{g}$  is isomorphic to one of  $(0, 0, 0, 0, 0, 0)$ ,  $(0, 0, 0, 0, 13 + 42, 14 + 23)$ ,  $(0, 0, 0, 0, 12, 14 + 23)$ ,  $(0, 0, 0, 0, 12, 34)$ ,  $(0, 0, 0, 0, 0, 12)$ .*

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- ▶ For a 6-dimensional nilpotent Lie group with an invariant complex structure  $J$ , the SKT condition is satisfied either by all invariant Hermitian metrics  $g$  or by none. There are 13 algebras that admit invariant KT structure but no invariant SKT structure.

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- ▶ For a 6-dimensional nilpotent Lie group with an invariant complex structure  $J$ , the SKT condition is satisfied either by all invariant Hermitian metrics  $g$  or by none. There are 13 algebras that admit invariant KT structure but no invariant SKT structure.
- ▶ By Malcev's Theorem above Lie groups  $G$  admit co-compact discrete subgroups  $\Gamma$ , so  $\Gamma \backslash G$  is nilmanifold.

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- ▶ By Malcev's Theorem above Lie groups  $G$  admit co-compact discrete subgroups  $\Gamma$ , so  $\Gamma \backslash G$  is nilmanifold.
- ▶ Similar calculations in dimension four give two algebras  $(0, 0, 0, 0)$ ,  $(0, 0, 0, 12)$  (precisely those with invariant complex structure).

Greater flexibility in SKT condition compared with Kähler implies lack of central results, e.g., have **no Calabi-Yau Theorem**. Best to date is:

### **Theorem (Tosatti & Weinkowe, 2009)**

*Let  $(M^{2n}, g, J)$  be a compact connected SKT manifold for which  $\omega^2$  is  $\partial\bar{\partial}$ -closed. For every smooth real function  $f$  on  $M$  there exists a unique smooth real function  $\phi$  on  $M$  solving*

$$(\omega + i\partial\bar{\partial}\phi)^n = \left( \frac{\int_M \omega^n}{\int_M e^f \omega^n} \right) e^f \omega^n, \text{ with}$$
$$\omega + i\partial\bar{\partial}\phi > 0, \sup_M \phi = 0.$$

*In particular  $\omega + i\partial\bar{\partial}\phi$  is an SKT form on  $M$  with the same volume as  $\omega$ .*

## SKT geometry and bi-Hermitian geometry

SKT structures frequently appear in terms of **bi-Hermitian structures**:  $M$  is a manifold endowed with SKT structures  $(g, J_+)$  and  $(g, J_-)$  such that

$$(\bar{\partial}_+ - \partial_+) \omega_+ + (\bar{\partial}_- - \partial_-) \omega_- = 0.$$

By [Apostolov & Gualtieri] the above is equivalent to having a generalized Kähler structure on  $M$ .

## Why strong KT geometry?

Quest for canonical choices of metric compatible with given complex structure.

- ▶ SKT **less restrictive** than being Kähler;
- ▶ SKT condition seems to **behave well**, e.g., blow-up of strong KT manifold at point or along compact complex submanifold again SKT;
- ▶ Good understanding of SKT geometry helpful in study of strong HKT geometry.

## Almost hyperHermitian structure

**Almost hyperHermitian manifold:**  $(M, g)$  Riemannian with three almost complex structures  $I, J, K$  satisfying:

- ▶  $K = IJ = -JI$ ;
- ▶  $g(X, Y) = g(IX, IY) = g(JX, JY) = g(KX, KY)$ .

Define  $\omega_I = g(I\cdot, \cdot)$ ,  $\omega_J = g(J\cdot, \cdot)$ ,  $\omega_K = g(K\cdot, \cdot)$ .

## HKT manifold

### Definition

An almost hyperHermitian manifold  $(M, g, I, J, K)$  is a **hyperKähler manifold with torsion**, briefly an **HKT manifold**, if

$$Id\omega_I = Jd\omega_J = Kd\omega_K. \quad (1)$$

## HKT manifold

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An almost hyperHermitian manifold  $(M, g, I, J, K)$  is a **hyperKähler manifold with torsion**, briefly an **HKT manifold**, if

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HKT condition implies integrability of  $I, J, K$ . Bismut connection for  $(g, I)$ ,  $(g, J)$ , and  $(g, K)$  coincide. Torsion three-form is

$$c = -Id\omega_I = -Jd\omega_J = -Kd\omega_K.$$

Have **strong HKT** if  $dc = 0$ .

## Example

Consider  $SU(3)$ . Choose following basis of its Lie algebra  $\mathfrak{su}(3)$ :

$$A_1 = i(E_{11} - E_{22}), \quad A_2 = i(E_{22} - E_{33})$$

$$B_{\ell m} = E_{\ell m} - E_{m\ell}, \quad C_{\ell m} = i(E_{\ell m} + E_{m\ell}) \quad (1 \leq \ell < m \leq 3).$$

Denote its dual basis by  $a_1, \dots, c_{23}$ . Minus Killing form gives inner product

$$\langle \cdot, \cdot \rangle = -a_1 a_2 + 2(a_1^2 + a_2^2 + b_{12}^2 + \dots + c_{23}^2).$$

Orthogonal (Joyce) decomposition

$$\mathfrak{su}(3) = \mathbb{R} \oplus \mathfrak{d} \oplus \mathfrak{f}.$$

$$\mathbb{R} = \text{span} \{A_1 + 2A_2\}, \quad \mathfrak{d} = \text{span} \{A_1, B_{12}, C_{12}\},$$

$$\mathfrak{f} = \text{span} \{B_{13}, B_{23}, C_{13}, C_{23}\}.$$

Define action of  $I, J, K$  by taking highest root  $\mathfrak{su}(2)$  ( $\mathfrak{d}$ ) letting it act on complement  $(\mathbb{R} \oplus \mathfrak{f})$ , e.g.

$$I \left( \frac{1}{\sqrt{3}}(A_1 + 2A_2) \right) = A_1, \quad I \text{ acts as } \frac{1}{2} \text{ad}_{A_1} \text{ on } A_1^\perp \cap \mathfrak{d}$$

$$\text{ad}_{A_1} \text{ on } \mathfrak{f}.$$

Left translating above gives almost hyperHermitian structure on  $SU(3)$ . Can verify that

$$\begin{aligned} -c &= Id\omega_I = Jd\omega_J = Kd\omega_K = a_1(2b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23}) \\ &\quad - a_2(b_{12}c_{12} - b_{13}c_{13} - 2b_{23}c_{23}) - b_{23}c_{12}c_{13} - b_{13}c_{12}c_{23} \\ &\quad - b_{12}c_{13}c_{23} - b_{12}b_{13}b_{23}. \end{aligned}$$

Gives **strong HKT** structure on  $SU(3)$ .

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## Motivation

Classification of **invariant SKT structures** on six-dimensional nilpotent Lie groups. **Solvable Lie groups?**...Start in dimension four.

## Invariant SKT structure

Identifications:

- ▶  $\mathfrak{g}$  left-invariant vector fields on  $G$ ;
- ▶  $\mathfrak{g}^*$  left-invariant one-forms;
- ▶  $da(X, Y) = -a([X, Y])$ ,  $a \in \mathfrak{g}^*$ ,  $X, Y \in \mathfrak{g}$ .

Left-invariant almost Hermitian structure on  $G$  determined by inner product  $g$  on  $\mathfrak{g}$  and  $g$ -orthogonal linear endomorphism  $J: \mathfrak{g} \rightarrow \mathfrak{g}$  with  $J^2 = -1$ . **SKT condition** requirement that

- ▶  $J$  be integrable,  $d\Lambda^{1,0} \subseteq \Lambda^{2,0} + \Lambda^{1,1}$ ;
- ▶  $d\omega(J\cdot, J\cdot, J\cdot)$  is in the kernel of  $d$ .

## Solvable Lie algebras

$\Lambda^* \mathfrak{g}^*$  exterior algebra on  $\mathfrak{g}^*$  and  $\mathcal{I}(A)$  ideal in  $\Lambda^* \mathfrak{g}^*$  generated by subset  $A$ .  $\mathfrak{g}$  **solvable** means (dually):

### Lemma

*A finite-dimensional Lie algebra  $\mathfrak{g}$  is solvable if and only if there are maximal subspaces  $\{0\} = W_0 < W_1 < \cdots < W_r = \mathfrak{g}^*$  such that*

$$dW_i \subseteq \mathcal{I}(W_{i-1})$$

*for each  $i$ .*

$\mathfrak{g}$  solvable Lie algebra of dimension four.  $(g, J)$  an integrable Hermitian structure on  $\mathfrak{g}$  then there is orthonormal set  $\{a, b\}$  in  $\mathfrak{g}^*$  such that  $\{a, Ja, b, Jb\}$  is basis for  $\mathfrak{g}^*$  and either

- ▶  $\mathfrak{g}$  has **structural equations**

$$da = 0, d(Ja) = x_1 aJa,$$

$$db = y_1 aJa + y_2 ab + y_3 aJb + z_1 bJa + z_2 JaJb,$$

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**or**

$$da = 0, d(Ja) = x_1 aJa + x_2(ab + JaJb) + x_3(aJb + bJa) + y_2 bJb,$$

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- ▶ In first case,  $\{a, Ja, b, Jb\}$  may be chosen orthonormal and  $\omega = aJa + bJb$ , omitting  $\wedge$  signs.
- ▶ In latter case  $\omega = aJa + bJb + t(ab + JaJb)$  for some  $t \in (-1, 1)$ .

Above necessary, not sufficient. Need also

- ▶  $d^2 = 0$ ;
- ▶  $d(b - iJb)^{2,0} = 0$ ;
- ▶ SKT condition.

Impose unpleasant constraints on coefficients in structural equations...

...helps to observe that Lie's Theorem implies:

### Lemma

*A finite-dimensional Lie algebra  $\mathfrak{g}$  is solvable if and only if its derived algebra  $\mathfrak{g}'$  is nilpotent.*

So in dimension 4 we know  $\mathfrak{g}'$  is either **Abelian** or  $\mathfrak{h}_3$ .

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So in dimension 4 we know  $\mathfrak{g}'$  is **either Abelian or  $\mathfrak{h}_3$** . Now structural equations can be analysed, and we obtain:

### Theorem (TBM & Swann, 2009)

*Let  $G$  be a simply-connected four-dimensional solvable real Lie group. Then  $G$  admits a left-invariant SKT structure if only if its Lie algebra  $\mathfrak{g}$  is listed in the following table.*

$\mathfrak{g}'$	$\mathfrak{g}$	dim	$\pi_0$	Kähler	$(b_1 \dots b_4)$
$\{0\}$	$\mathbb{R}^4$	0	1	✓	(4, 6, 4, 1)
$\mathbb{R}$	$\mathbb{R} \times \mathfrak{h}_3$	0	1	×	(3, 4, 3, 1)
	$\mathbb{R} \times \mathfrak{r}_{3,0}$	1	1	✓	(3, 3, 1, 0)
$\mathbb{R}^2$	$\mathbb{R} \times \mathfrak{r}'_{3,0}$	1	1	✓	(2, 2, 2, 1)
	$\text{aff}_{\mathbb{R}} \times \text{aff}_{\mathbb{R}}$	2	1	✓	(2, 1, 0, 0)
$\mathbb{R}^3$	$\mathfrak{r}'_{4,\lambda,0} (\lambda > 0)$	1	2	✓	(1, 1, 1, 0)
	$\mathfrak{r}_{4,-1/2,-1/2}$	1	1	×	(1, 0, 1, 1)
	$\mathfrak{r}'_{4,2\lambda,-\lambda} (\lambda > 0)$	1	2	×	(1, 0, 1, 1)
$\mathfrak{h}_3$	$\mathfrak{d}_4$	2	1	×	(1, 0, 1, 1)
	$\mathfrak{d}_{4,2}$	2	1	✓	(1, 1, 1, 0)
	$\mathfrak{d}'_{4,0}$	2	1	×	(1, 0, 1, 1)
	$\mathfrak{d}_{4,1/2}$	1	1	✓	(1, 0, 0, 0)
	$\mathfrak{d}'_{4,\lambda} (\lambda > 0)$	1	1	✓	(1, 0, 0, 0)

## Remarks/Consequences

### Corollary

*There are four-dimensional solvable complex Lie groups whose family of compatible invariant Hermitian metrics contains both SKT and non-SKT structures.*

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### Corollary

*The four-dimensional solvable Lie algebras  $\mathfrak{g}$  that admit invariant complex structures but no compatible invariant SKT metric are:*

$\mathbb{R} \times \mathfrak{r}_{3,1}$ ,  $\mathbb{R} \times \mathfrak{r}'_{3,\lambda>0}$ ,  $\mathfrak{aff}_{\mathbb{C}}$ ,  $\mathfrak{r}_{4,1}$ ,  $\mathfrak{r}_{4,\mu,\lambda}$  ( $\mu = \lambda \neq -\frac{1}{2}$  or  $\mu \leq 1$ ),  
 $\mathfrak{r}'_{4,\mu,\lambda}$  ( $\lambda \neq 0, -\mu/2$ ),  $\mathfrak{d}_{4,\lambda}$  ( $\lambda \neq \frac{1}{2}, 2$ ),  $\mathfrak{h}_4$ .

## Remarks/Consequences

### Corollary

*Each unimodular solvable four-dimensional Lie group  $G$  with invariant SKT structure admits a compact quotient by a lattice.*

### Corollary

*The only four-dimensional solvable Lie algebra that is strong HKT is  $\mathbb{R}^4$ , which is hyperKähler. The algebra  $\mathfrak{d}_{4,1/2}$  admits both HKT and SKT structures; these structures are distinct. The remaining HKT algebras  $\mathfrak{aff}_{\mathbb{C}}$  and  $\mathfrak{r}_{4,1,1}$  do not admit invariant SKT structures.*

## Strategy for classifying SKT

Similar procedure expected to work in dimension six. But how do we attack classification of strong KT manifolds in general?

Are there any **tools** we can use?

### Corollary

*Each invariant SKT structure on a four-dimensional solvable Lie group  $G$  is invariant under a two-dimensional Abelian subgroup  $H \leq G$ .*

Such a fibration can be investigated using so-called **multi-moment maps** (work in progress).

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## Usual moment map

$(M, \omega)$  a **symplectic manifold**.

- ▶ Lie group  $G$  acts on  $M$  preserving symplectic form. Under appropriate circumstances, e.g.,  $H^1(\mathfrak{g}) = 0 = H^2(\mathfrak{g})$ , exists a (unique) equivariant map

$$\mu: M \rightarrow \mathfrak{g}^* \quad \text{s.t.} \quad d\langle \mu, X \rangle = \xi_X \lrcorner \omega, \quad (X \in \mathfrak{g}).$$

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Such a **moment map** is useful, e.g., in Kähler geometry, since fundamental two-forms are symplectic forms. For SKT manifolds  $c^B \neq 0$  in general, so not symplectic. But  $dc^B = 0$ .

Can  $\omega$  in moment map construction be replaced by closed invariant three-form  $c$  on  $M$ ?...

## Covariant moment maps

### Definition (Cariñena et al, 1991)

Let  $M$  be a manifold and  $\alpha$  a closed  $p$ -form on  $M$ . Let  $G$  be a Lie group acting on  $M$  preserving  $\alpha$ . A **covariant moment map** is a section  $\sigma \in \Gamma(\mathfrak{g}^* \otimes \Lambda^{p-2} T^* M)$  satisfying

$$d\langle \sigma, X \rangle = \xi_X \lrcorner \alpha \quad (X \in \mathfrak{g}).$$

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- ▶ Consistent with usual notion of moment map when  $p = 2$ . For  $p = 4$  fits with notion of quaternion-Kähler moment map;

## Covariant moment maps

### Definition (Cariñena et al, 1991)

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- ▶ Consistent with usual notion of moment map when  $p = 2$ . For  $p = 4$  fits with notion of quaternion-Kähler moment map;
- ▶ Case  $p = 3$  of interest. Do not want section as above, but (equivariant) map  $M \rightarrow V_{\mathfrak{g}}$  into vector space depending only on  $\mathfrak{g}$ .

## Calculation

$X, Y \in \mathfrak{g}$  s.t.  $[X, Y] = 0$  then

$$0 = [\xi_Y, \xi_X] \lrcorner c = d(\xi_Y \lrcorner \xi_X \lrcorner c),$$

so if  $b_1(M) = \dim H^1(M, \mathbb{R}) = 0$ , we have a scalar function  $\nu_{X,Y}: M \rightarrow \mathbb{R}$  satisfying

$$d\nu_{X,Y} = \xi_Y \lrcorner \xi_X \lrcorner c.$$

More generally consider  $p = \sum_j X_j \wedge Y_j \in \mathcal{P}_{\mathfrak{g}} = \ker(L: \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g})$ , thus  $0 = L(p) = \sum_j [X_j, Y_j]$ . We get as before  $\nu_p: M \rightarrow \mathbb{R}$  such that

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Module  $\mathcal{P}_{\mathfrak{g}}$  so natural that one would expect much is known...

## Remark on module $\mathcal{P}_{\mathfrak{g}}$

...seems not to be the case. For instance have not found explicitly the following result (which follows from result of Wolf on isotropy irreducible spaces):

### Theorem

*Let  $\mathfrak{g}$  be a compact simple Lie algebra. Then  $\Lambda^2 \mathfrak{g}$  splits into the following irreducible modules*

$$\Lambda^2 \mathfrak{g} = \mathfrak{g} \oplus \mathcal{P}_{\mathfrak{g}}.$$

## Multi-moment maps

### Definition (TBM, 2008)

Let  $M$  be a manifold and  $c \in \Omega^3(M)$ ,  $dc = 0$ . Let  $G$  be a Lie group acting on  $M$  preserving  $c$ . A **multi-moment map** is an equivariant map  $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$  such that

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Have thus way obtained (appropriate) equivariant map from  $M$  into representation  $\mathcal{P}_{\mathfrak{g}}$ ...if multi-moment maps exist...

## General existence

### Theorem (TBM, 2009)

Let  $M$  be a connected manifold and  $c \in \Omega^3(M)$ ,  $dc = 0$ . Let  $G$  be a Lie group acting on  $M$  and preserving  $c$ . A multi-moment map  $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}$  exists if one of the following conditions is satisfied:

- ▶  $b_1(M) = 0$  and  $G$  is **compact**;
- ▶  $M$  is **compact** orientable manifold with  $b_1(M) = 0$ , and  $G$  preserves a volumeform on  $M$ ;
- ▶  $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$ .

Another frequently encountered situation, where moments map exist:  $M$  is simply-connected and  $G$  is **Abelian**.

## Example 1

$N$  a manifold.  $M = \Lambda^2 T^*N$  has canonical two-form  $\beta$ . At  $\alpha \in M$  define

$$\beta_\alpha = \pi^*(\alpha), \quad \pi: M \rightarrow N \text{ is the bundle projection.}$$

Lie group action  $G \times N \rightarrow N$  lifts to action  $G \times M \rightarrow M$  preserving  $\beta$  and hence the closed three-form  $c = d\beta$ . A multi-moment map  $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$  is given by

$$\langle \nu, \rho \rangle = \beta(\xi_\rho).$$

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$$\langle \nu, \rho \rangle = \beta(\xi_\rho).$$

Note similar construction exists in symplectic setting (and explains the name "moment" map).

## Example 2

Consider the Lie group

$$G = \left\{ \left( \begin{array}{cccc} 1 & 0 & ye^{-x} & w \\ 0 & e^x & 0 & y \\ 0 & 0 & e^{-x} & z \\ 0 & 0 & 0 & 1 \end{array} \right) : x, y, z, w \in \mathbb{R} \right\}$$

with corresponding Lie algebra  $\mathfrak{d}_4$ . For one concrete choice of invariant (non-Kähler) SKT structure  $(g, J)$  on  $G$  we find an Abelian group  $A$  acting on preserving the SKT structure such that there are maps  $\nu_J, \nu: G \rightarrow \mathbb{R}$  given by

$$\nu_J = -\exp(-x), \quad \nu = z,$$

which are  $A$ -invariant and satisfy

$$d\nu_J = \xi_W \lrcorner \xi_Y \lrcorner d\omega, \quad d\nu = \xi_W \lrcorner \xi_Y \lrcorner Jd\omega.$$

These are multi-moment maps.

## Applications

- ▶ Take  $M = SU(3)$  with Joyce SHKT structure.  
 $G = SU(3) \times U(1)$  acts on  $M$  preserving SHKT structure.  
Exists multi-moment map  $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$  with image  $\mathbb{C}P(2)$ .  
Possible to **reconstruct the hypercomplex** structure on  $M$   
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- ▶  $(M^7, \phi)$  is a  $G_2$ -manifold with an  $S^1$ -symmetry. Using Kähler  
reduction it has been shown that one sometimes get  $M$  as a  
 $T^2$  bundle over a hyperKähler manifold. This **reduction** is  
more naturally described in terms of multi-moment maps.

## Work to be done

Work in progress/to be done:

- ▶ use multi-moment maps as tools in **classification of (primarily) strong KT and strong HKT** manifolds;
- ▶ study **orbits**  $\mathcal{O} \subset \mathcal{P}_{\mathfrak{g}}^*$ ;
- ▶ study **quotient constructions**;

## References

- ▶ P. Gauduchon, *Hermitian connections and Dirac operators*, *Unione Matematica Italiana. Bollettino. B. Serie VII*, 11, 1997, 2, suppl., 257–288.
- ▶ P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, *Mathematische Annalen*, 267, 1984, 4, 495–518.
- ▶ A. Fino, M. Parton, S. Salmon, *Families of strong  $KT$  structures in six dimensions*, *Commentarii Mathematici Helvetici*, 79, 2004, 2, 317–340.

## References

- ▶ V. Apostolov, M. Gualtieri, *Generalized Kähler manifolds, commuting complex structures, and split tangent bundles*, Comm. Math. Phys., 271, 2007, 2, 561–575.
- ▶ T. Madsen, A. Swann, *Invariant strong KT geometry on four-dimensional solvable Lie groups*, arXiv:0911.0535v1 [math.DG].
- ▶ J. Cariñena, M. Crampin, L. Ibort, *On the multisymplectic formalism for first order field theories*, Differential Geom. Appl., 1, 1991, 4, 345–374.